Chapter 6

Flattened: How Gauss and Legendre Conquered the Data

What does it mean to be flattened? Is it a literal transformation of a shape? Is it a rendez-vous with hardship that leaves one incapacitated? Is it an exposure of a lie that is laid flat and bare? Is it the discovery of a fundamental truth that is laid flat and bare? In this chapter, it is all of these. All of these are landmarks that have led us to our current encounter with AI.

6.1 Newton, from his Desk

From the cloistered confines of his office, Newton deciphered the universe. He proclaimed the three Laws of Motion, the Law of Gravity (inverse square law), and developed calculus. Calculus allowed Newton to quantify outcomes that result from the laws. Newton proved that the laws of the universal traffic cop binds the planets to travel along Kepler's ellipse.

Not only did Newton apply his methods to the heavens, he applied it to the Earth as well. Long before Christopher Columbus' journey our fore bearers dismissed the notion of a flat Earth. Ancient Greek and Chinese astronomers[1] observed the change in declination of the stars as an observer moved from North to South. Looking to the Sun and Moon for inspiration, they reasoned that the Earth is round.

Once deciding upon a shape, a natural question is, what is its size? In 240 B.C. the Chief Librarian at the Library of Alexandria set about answering this question. Eratosthenes measured the shadow length of a gnomon on the library grounds. The Chief Librarian was aware that at the same time he took his measurement, the Sun was directly overhead and casting no such shadow in a city Syene, 5,000 stadia to the south. It was on the day of the summer solstice and Syene was on the boundary of the Sun's annual northerly sojourn.

From his gnomon measurement, Eratosthenes deduced that the angular separation between Syene and Alexandria was 7.2° equivalently $1/50^{th}$ of a circle. Multiplying the distance between the cities by 50 , Eratosthenes arrived at 250,000 stadia as the circumference of the Earth. A conversion of the stadia to the kilometer yields 39,400 kilometers which is remarkably close to today's estimate of the equator's circumference, 40,075 kilometers.

Eratosthene's shadow shed light upon the very body that we walk upon. Nineteen centuries later, with Britain embarking upon its colonial experiment and reshaping international politics, Newton would reshape the Earth.

[1] Forgive our ignorance. Other cultures probably shared the awareness of the Greek and Chinese peoples.

Just as nature's laws bind the planets to their elliptical orbits, so would the laws force their configuration into specified shapes. A perfect sphere was the configuration for a lazy, stationary body, but for the Earth that frantically spins like a top, Newton recognized that the laws would bind the Earth to a configuration that would squeeze the sphere.

An individual standing still at the equator is actually orbiting the center of the Earth at a dizzying speed of 1,670 kilometers per hour (1040 miles per hour). The spin causes a centrifugal force on the man that opposes the inward force of gravity.

So let's take an imaginary journey while seated at our desk. Suppose the Earth is spherical. An individual at the north pole stands on a typical bathroom scale (spring compression). The reading is two hundred pounds and the individual feels a bit overweight. That same individual takes the scale to the equator and once more stands on the scale. This time the reading is 199.3 pounds. Should the individual be happy with the weight reduction?

No!!! Emphatically no. The individual's mass is the same in both locations. At the equator the body mass pushes against the scale a bit more lightly because the centrifugal force caused by his orbital speed about the Earth's center, counterbalances gravity. The effect of gravity is greater at the poles than it is at the surface.

Newton saw this clearly and recognized this has implications for the Earth's shape. If gravity's squeeze is greater at the poles, then gravity squeezes the poles closer to the center of the Earth and the Earth bulges at its equator. We live on a dumpy Earth that Newton declared is close to an oblate spheroid[2].

Using calculus Newton went on to make some suppositions about about the Earth's homogeneity and concluded that the flattening coefficient is $f = 1/230$. The modern day value as determined by physical measurements is $1/298.257$.

What is the flattening coefficient? The formula below defines it.

$$
\begin{aligned}
f &= \frac{R_E - R_P}{R_E} \\
&= 1 - \frac{R_P}{R_E}
\end{aligned}
$$

Where

- R_E is the distance from Earth's center to the equator
- R_P is the distance from Earth's center to the poles

For a perfect sphere, $R_E = R_P$, therefore the flattening coefficient is zero. Newton's value of $1/230$ is very small and the today's value is smaller still. The main cause of the discrepancy between Newton's value and today's value is Newton's supposition of homogeneity of mass distribution. The assumption eases computational effort and Newton was well aware of its shortcomings. His method is an approximation that correctly shows a small flattening coefficient that yields an oblate spheroid.

The following sections examine efforts to refine Newton's estimate.

[2] An oblate spheroid has an elliptic cross section with the minor axis as the polar axis. Rotating the ellipse about its minor axis gives the oblate spheroid.

6.2 Lean vs Dumpy

Cartographers do not have the luxury of theorizing in comfort. They must subject Earth to their measurements. But Earth in turn forces the surveyor to submit to all deprivations imaginable, and some not so imaginable.

In 1670 Loius XIV commissioned Giovanni Cassini to map all of France. Cassini's son, grandson, and great grandson would take on the project. The families had tight bonds. In 1783, Louis' great great grandson, Louis XVI, offered Giovanni's great grandson, Jean-Dominique a commission to finish off the project and direct the French side of the Anglo-French survey. While Jean Dominique initially accepted, he withdrew soon after the work began. Jean Dominique did lend support to the publication of the report, however publication awaited 1793. This was in the midst of the French Revolution's reign of terror; neither Louis XVI nor Jean Dominique could celebrate the publication from their prisons. Unfortunately, that same year, a brutal fate that left him headless awaited Louis XVI, so he never saw the final product of the mapping project. Jean Dominique would live to the enviable age of 97 passing on in 1845.

The initial survey commissioned in 1670 fell under the purview of the four yield old French Academy of Sciences. One might guess that such a survey would be viewed favorably by the King. Wanting to show its value, the Academy proposed the project. This began a long tradition of royal support for survey expeditions.

In 1720, Jacques Cassini, the son of Giovanni, published results of a survey that he performed along a meridian running through Paris. The data from his survey indicated that Newton, got things backwards. The Earth is lean; the distance from pole to pole is greater than the equator's diameter. The publication stirred a great debate among the scientific community and members of the Academy, Newton's oblations versus Cassini's prolations.

The debate split along professional and political alliances. The prestigious and well connected Cassini family had its supporters. Many of the more theoretical mathematical community aligned themselves with Newton. Among the oblations were Clairaut[3] and Maupertuis.

The prolations believed that theory must confront reality, and Cassini's measurements presented reality. In opposition was the argument that while Cassini took great care, the oblateness of the Earth was far too subtle for Cassini to be able to measure across the small segment of the Earth near Paris. To align theory with reality, would require measurements of meridional segments near the equator and near one of the poles. It was agreed, the Academy would commission two expeditions, one to Peru and one to Lapland. Each would take measurements that would resolve the debate.

Although not a scientist and not a member of the Academy, among those in Newton's camp was Voltaire, France's preeminent philosopher. At one time in their lives, he and Maupertuis had been friends. These two brilliant men had a fallout, most likely over something stupid.

6.3 To the Ellipse's Edge

Lapland

In the summer of 1736, a group of French scientists set off not for the salons of Paris or the libraries of the Academy, but for the edge of the known world—the Arctic Circle. Their destination: Tornedalen, a sparsely populated region of northern Lapland (modern-day Finland). Their mission: measure the length of a degree of latitude near the pole to help settle a bitter scientific argument.

[3] Clairaut furthered Newton's research and considered the inhomogeneous case where density of the rotating body is not uniform. In opposition to Newton's belief, Clairaut came to the conclusion that the flattening coefficient of the body with uniform density is greater than the flattening coefficient with nonuniform density. Clairaut got it right. On occasion, even Newton made an error.

Leading the team was the celebrated polymath Pierre-Louis Moreau de Maupertuis. Alongside him were Alexis Clairaut, Charles Le Monnier, the team's astronomer, Abbé Outhier, a clergyman and cartographer who kept a detailed diary, and Anders Celsius, the Swedish scientist and local guide. They were Paris-trained minds in powdered wigs, navigating the snowdrifts of Lapland on reindeer-drawn sleds.

The scientists were hosted by the Sámi people, whose skill in cold-weather survival proved far superior to that of the philosophers. The French, unfamiliar with Arctic travel, took repeated spills from their sledges. Outhier wrote of "violent falls" and "racing sleds that refused to slow", and one misadventure in which a reindeer, startled by wind, bolted with a cart full of sextants. "The beast," Maupertuis dryly noted, "has no sense of reverence for the laws of gravitation."

The weather was brutal, and much of the measuring had to take place at night, in subzero temperatures. Ink froze in the quill before it could touch paper. Instruments warped, and thermometers shattered in the cold. Despite these conditions, Clairaut calculated trigonometric solutions under the aurora-lit sky while Maupertuis took angular measurements atop icy hills.

In moments of reprieve, there were odd diversions. The team held impromptu reindeer races—the Sámi, of course, always won. There were murmurs (possibly imagined) that Maupertuis had formed a romantic attachment to a local Sámi woman, though this may have been no more than Enlightenment-era gossip, exaggerated by Parisian wits looking for scandal in the snow.

Yet beneath the humor, the mission was serious—and successful. After months of painstaking measurement, the team determined that one degree of latitude in Lapland was longer than one measured in southern France by Jacques Cassini. This was the key result: the Earth is flatter at the poles, just as Newton had predicted.

Maupertuis rushed home and quickly published his results, declaring victory for the Newtonians. He dubbed Lapland "a desert of snow and rock," but claimed it had given the world one of the most important confirmations of modern science.

Meanwhile, in the jungles of Peru, La Condamine was still toiling on the equatorial expedition. La Condamine would later criticise Maupertuis for pronouncing a final verdict prior to awaiting for the data from Peru. This seems to have had no effect upon Maupertuis, but stinging words from another source would embed itself in history's record book. Voltaire mocked Maupertuis, "Maupertuis went to the ends of the Earth only to affirm that which Newton deduced from his desk."

Peru

If science in Lapland was a test of endurance in snow and silence, science in the Andes was an exercise in survival at the intersection of fire and ice. The French expedition to what is now Ecuador, launched in 1735, remains one of the most grueling scientific journeys of the Enlightenment-nature tested equally the minds and bodies of the expedition force.

Led by Charles-Marie de La Condamine, and joined by Pierre Bouguer and Louis Godin, the expedition aimed to measure one degree of latitude near the equator, completing the global experiment to determine the shape of the Earth. But between the equator and their measuring stations lay a gauntlet of fevers, frostbite, avalanches, and betrayal.

The ordeal began in the sweltering lowlands near Guayaquil, where the team had to transport their precious scientific instruments—quadrants, zenith sectors, pendulums—through mosquito-infested jungles and flooded forest paths. The humidity warped wooden casings; leather straps rotted, and ants gnawed at the crates. "We were not sure," La Condamine quipped with dry resignation, "whether the insects would eat the instruments before they ate us." His remark, equal parts complaint and gallows humor, captured the absurdity of trying to preserve Enlightenment precision amid tropical decay.

Yet worse was still ahead. As they ascended the Andes, heat gave way to altitude, thin air, and the cruel paradox of equatorial snow. The same team that had sweated through the jungle now faced freezing gales atop Mount Chimborazo, Antisana, and Cayambe, measuring baselines in glacier fields at 15,000 feet. They bivouacked on icy ridgelines where like their counterparts in Lapland, ink froze in its bottle and the cold shattered thermometers. By day, the sun roasted one side of their faces; by night, they wrapped themselves in saddlebags and tried to sleep beneath snow-covered rocks.

The terrain was one challenge; a challenge they compounded by squabbling among themselves. Godin and Bouguer, once collaborators, became bitter rivals. Disagreements over data, procedure, and credit led to open hostility. The team fractured—duplicating measurements independently rather than cooperating. Their scientific fidelity never wavered, but their friendships froze even faster than the mountain air.

Through all this, they persisted. They were frequently slowed by Spanish colonial bureaucracy, occasionally harassed by suspicious officials, and routinely misled by poor maps and roads that were little more than obstacle courses. Two assistants died—one of disease, the other from exposure. Godin's wife, left in France, would later brave the Amazon in a doomed attempt to reunite with him. By the time the expedition ended, nine years had passed, and what remained of the team was physically diminished and personally estranged.

And yet—they succeeded. The measurements showed that a degree of latitude at the equator is shorter than in France or Lapland. The Earth, as Newton predicted and Cassini denied, is flattened at the poles. The argument was over, but more battles lie ahead.

Discord among the members of the Peruvian expedition caused distrust in their data. Using the Lapland and Cassini data, Clairaut updated Newton's estimate of the flattening coefficient lowering it from $1/230$ to $1/170$. Earth was even dumpier than before. However, this would not be the final word; the secrets from Peru extracted with such hardship would not remain silent forever.

As for La Condamine, he returned to France with his journals and no shortage of stories. His writings mixed geometry with jungle drama, filled with scientific results and sharp literary wit.

6.4 The Meter

In the eighth decade of the 18^{th} century, grandiose ideas swirled about the French Literati. The American Revolution awoke a spirit of republicanism. Adam Smith's *The Wealth of Nations* soaked into the intellectual soil of France fueling free market philosophies that many modern day economists view as doctrine. Newton's calculus and physics inspired investigations into areas that were previously impossible to explore. And while on the subject of exploration, the French Academy of Sciences pitched their grandiose idea of an international system of unified measurement to Louis XVI, and he liked it. This would launch another grand adventure.

The advantages of a standardized measurement system are obvious and need not be mentioned. The quickest route to a solution might also be obvious; among the thousands of measurement standards in place in France at the time, select the most commonly used one and declare that to be the national standard. As for an international standard, well just take a first step and create a national standard.

But the first step was not grandiose. The intention of the members of the Academy was a new set of measurement standards with the end goal in mind; it must be an international standard. The solution, establish a measurements based upon nature which is owned by none, but used by all. After much debate, the Academy decided that the unit of length would be $1/10,000,000^{th}$ the distance from the equator to the North pole. This was a permanent length for a permanent standard.

This seems acceptable. The obvious next step is to make the calculation. After all, enough data was available from all of the surveying expeditions so that the skilled mathematicians of the Academy could calculate a very good approximation. In fact, a calculation was made and it was very good. But this was not acceptable for such a grandiose project. It must not be a mere approximation based upon past measurements; new measurements with the most up to date methods and equipment must assure perfection.

There were those who brought up practical issues. Prominent among the doubters was Antoine Lavoisier who had some weighty arguments. The certainty that the Earth's circumference is an eternally fixed number was called into question. Additionally, perfection is unattainable. There will certainly be some degree of measurement error. Also, in the not too distant future, new tools and new methods will be available rendering the outcome of any previous survey less accurate. Are we to redefine the standard every time a new process can improve upon the measurement?

The excitement of the search for perfection overruled the objections. In 1792, Louis XVI signed over two dozen letters from the King informing the readers that they should provide all manners of assistance to the holders. The letters were placed in official envelopes that were bound by the royal seal. The northern expedition leader, Jean Delambre, and southern expedition leader, Pierre Mechain, received their allotment of letters which would provide royal protection as they proceeded toward one another from their initial positions, Dunkirk and Barcelona respectively. Rather than guaranteeing safety, the letters were nearly the cause of death.

The Technique

The method that underlies the surveys undertaken since Cassini[4] is quite simple[5]. Proposing such a method could well have been a homework exercise for the students of Euclid attending his lectures in the third century B.C.E.. Mark one triangle by three easy to spot points within the survey area. Church steeples were favored marks. In remote areas a platform atop a hill with a flag pole may be constructed.

With three marking sites available, measure two of the angles and the length of one side of the triangle. The third angle and the lengths of the two sides can be determined through a simple, well known computation. In practice, as a quality check on the angular measurements, the surveyor measures all three angles and assures that they sum to $180°$.[6]

Extend the surveyed area by identifying a marker in a favorable position. Along with two of the previously established marks, a new triangle is available. Furthermore, one side of the new triangle is also a side of the previous triangle whose length has been established. Measuring the angles of the new triangle allows one to calculate the length of the remaining two sides. In this manner one can construct a network of triangles that covers the area of interest and calculating the lengths of the sides of every triangle while measuring the length of only one side in the entire network.

As a further quality check, one might measure the length of a side of a triangle that is distant from the original triangle and compare it with the calculated length. How satisfying it is when the measurement and calculated length align.

[4]Surveys centuries earlier in China use the same method.

[5]This is a simplified explanation that ignores the spheroidal geometry of the Earth. Configuring latitude , longitude, and distance to match a spheroidal shape is not so simple.

[6]Once again, spheroidal geometry introduces complexities; on spheroids the angles of a triangle do not in general sum to 180 degrees.

The Baseline

Success of the entire enterprise demands a highly accurate measurement of one side of a single triangle known as the baseline. The baseline measurement determines the calculated measurements of every side of every triangle in the network. An error here infects the entire network of triangles. Note that it is unnecessary to measure a baseline at the outset. One can establish the triangles and make all the angular measurements prior to selecting a baseline and making the baseline measurement.

Given the criticality of the baseline measurement, it is a fixation among surveyors. The French approach reflects a neurotic obsession. The team of surveyors would have rods of equal length that they would lay one in front of the other. Counting the total number of rods required to cover the baseline would reveal its length – a simple idea for a child, a complex process for members of the Academy.

The first complication was assuring the rods were all of equal size. Then there is the possibility of corrosion and thermal expansion that might alter the length of the rods. Prior to the meter, the measurement length adopted by the Academy was the toise. The most skilled craftsman of France, Etienne Lenoir, produced the rods of a platinum-iron alloy, each two toises in length. The special alloy was corrosion resistant and less susceptible to thermal expansion than other known materials.

Then there is the complication of assuring the perfect alignment of the rods over uneven terrain. The surveyors used special sighting devices, plumb lines , and wooden trestles to assure the horizontal placement of the rods, abutted with one another in perfect alignment with the baseline's endpoints.

Operations were as follows. A train of perhaps two to four end to end rods was initially aligned. Temperature readings were taken so that length accounting for thermal expansion could be performed. Once surveyors recorded their entries, they removed the caboose of the train and placed it in the lead position. Realignment, temperature recordings and quality check were all necessary. A single repositioning might take a half hour. The surveyors repeated this operation along the entire baseline.

The Peruvian expedition had the misfortune of encountering terrain that resembled the aftermath of a heavily bombarded battlefield. It took them over six months to finalize their measurement of the 13,152 toises long baseline (25.6 km). At two toises per repositioning, the team had to relocate the caboose around 6,571 times.

By contrast, Delambre's northern baseline was 6,075.90 toises in length (11.8 km), 3,038 caboose exchanges, and it took 33 days to make the measurement, averaging a bit more than 350 meters per day. At typical walking speed, one can cover the distance in four minutes. Delambre measured an additional baseline in southern France. Computations of the the lengths of triangle sides in the combined network of Delambre and Mechain using each baseline matched demonstrating the precision of their observations; maybe everything was a bit too tidy.

The Finest Instrument

In 1783, while the British and French were at war on opposing sides of the American Revolution, King Louis XVI and the Royal Society of London agreed upon a joint project to survey the lands from Paris to London. The project would establish the meridional difference between the two cities.

A reliable survey requires accurate measurement of all angles within the network of triangles covering the survey area. The British commissioned Britain's most accomplished machinist, Jesse Ramsden, to design and construct a specialized theodolite that surveyors would use to measure the angles. Ramsden centered his instrument about a circular sextant around one meter in diameter (Of course the meter did not exist at this time, Ramsden used other units.)

The theodolite was impressively accurate. A gearing mechanism to reposition special siting devices allowed for precision alignment of the observing eye with the target. Its impressive size also enhanced the precision.

The British leader of the expedition William Roy, might have marveled at the impressive theodolite when he first saw it. After schlepping the 200 pound instrument through the difficult countryside, dragging it up hilltops, elevating it upon observation platforms, and carrying it to the top of church steeples, he had to have been jealous when he saw the French counterpart.

The French turned to Etienne Lenoir, who improved upon Jean Borda's invention of the repeating circle, the name given to the theodolite, so that it was more practical for use in the field. Borda saw that averaging of many observations over a sextant with a small radius would produce the same precision as a single measurement using a much larger sextant. Toward this end he fashioned a design that allowed the surveyor to take multiple observations one after the other in very quick succession. The method ingeniously allowed the surveyor to take only two recordings, not a recording for each measurment. This hastened the speed with which the surveyor could take multiple observations. The diameter of the repeating circle was less than one third the diameter of Ramsden's sextant.

Lenoir improved upon the design so that the instrument was a manageable 25 pounds. Not only that, he provided the French surveyors with two instruments. While the British wrestled with their 200 pound behemoth, The French nimbly criss-crossed the countryside with their two lightweight repeating circles. And the precision of the repeating circle was equal to that of the Ramsey theodolite. The French covered much more ground in shorter time than the British.

Leading the French team was Pierre Mechain. While other expeditions' reports identify assistants who were necessary for support, the report compiled from Mechain's records do not indicate the names of Mechain's assistants. That is strange.

The Machine and the Leader

A permutation of the letters in the name Mechain yields "Machine". Machine is an apt description of Mechain. He was an astronomer who investigated the heavens night after night after night... He had unequaled energy that allowed him to peer at the sky for hour upon hour and continuously take observations at a pace that would exhaust others. Using his formidable observation capacity, Mechain identified more comets than any of his predecessors or contemporaries.

One would be hard pressed to find an individual who was more suited to the night time occupation of observational astronomer than Mechain. He had a well deserved reputation among his peers. With the successful completion of the French side of the Anglo-French survey, Mechain was a natural candidate to lead the Dunkirk-Barcelona meridional survey. The appointment was forthcoming; Mechain was the senior member of the expedition. The leadership who made the appointment did so on the basis of Mechain's success as an individual contributor. His night time duties provided little opportunity for them to assess his character. If they had a more thorough vetting, perhaps they would have made another choice.

Eliminate the "m" and "b" from the Delambre, then from the remaining letters one can spell the word "Leader". Delambre was a highly capable mathematician and astronomer. Prior to his appointment on the Dunkirk-Barcelona meridional survey, he was best known for compiling the *Tables du Soleil (Sun Tables)* which became a standard reference in astronomy and navigation. The work highlighted both excellent observational skills (accuracy of observations) and computational skills. On the computational side, Delambre proved to be somewhat of a human computer, a skill highly prized and necessary for the computational efforts of large scale surveys.

For his publication of the *Tables du Soleil*, in 1786 the Academy awarded Delambre its annual *Grand Prix* *(grand prize)* and elected him to the prestigious Academy. His congenial collaboration with other members impressed everyone and he became second in charge of the Dunkirk-Barcelona meridional survey.

The Academy assigned Mechain the southern leg of the survey; he was to lead a team that would start in Barcelona and work its way north. The Academy assigned Delambre and his team the northern leg of the survey that included both Paris and Dunkirk. He was to work his way south and meet with Mechain. In this manner, designations of first and second in charge reflected Mechain's seniority within the Academy, not roles and responsibilities in the survey.

Delambre proved to be a capable leader. He mentored his team members, teaching them how to use the repeating circle and how to record observations. His team would independently enter observations into their logbook and validate one another's work; Delambre most likely approved of all entries. More importantly, his manner and social skills most likely saved his life and that of his teammates.

On the contrary, Mechain did not permit anyone other than himself to operate the repeating circle; he made all observations. Beyond denying access to the repeating circle, Mechain further distanced his subordinates from the project by denying them access to the logbook. Forget the possibility of making or reviewing entries, Mechain's team could not even have a peek at the logbook. The repeating circle and logbook were Mechaine's personal possessions. His subordinates carried equipment, set up viewing and siting platforms, essentially performing all of the grunt work. (Extenuating circumstances caused a one time exception to this rule. More on that below.) Perhaps had he been more collaborative, Mechain could have avoided the misfortune that awaited him.

Close Call in the North

After two months of searching for three points that would establish his first triangle and setting up a citing platform atop one of the sites, Delambre was ready to take his first observation. It was night time, August 10, 1792 in the town of Montjay just north of Paris. Delambre was in a church steeple. One of the siting points was Montmarte in Paris. Unable to pinpoint his target during the daytime, Delambre sent a team member to light a flare at the target location on Montmarte.

The previous weeks saw many disappointments and delays caused by unexpected difficulties. Delambre assumed he would be further north on his way to Dunkirk. Finally he could take the first observation of his first triangle. But there was no signal from Montmarte, so the anticipation of being able to take a measurement gave way to another disappointment. From the church steeple in Montjay, Delambre did however observe a fire around the Tuileries Palace.

Unknown to Delambre, the fire announced the end of Louis XVI's rule and the beginning of the revolutionary government. The revolution announced itself to Delambre the next day when citizens confronted and effectively arrested the members of the expedition. Delambre's explanation that the telescopes mounted on the repeating circle were not spying instruments in support of a Prussian army that the King had called upon, but instead surveying instruments meant to measure the Earth's girth did not overcome the citizens' suspicion. Delambre could read the crowd's reaction; why don't you try to sell me valuable land to the east of Dunkirk while you are at it?

Things went no better when Delambre displayed passports with their royal certification. This merely engendered more suspicion as the passports were further evidence that the the spies were in cahoots with the King. Showing his capacity to maintain his wits, Delambre was able to persuade the citizens that the matter should be brought to the mayor's attention, rather than rush to hasty judgment and take matters in their own hands. And so the citizens escorted Delambre and his team to meet with the mayor. That night, with the assistance of the mayor, Delambre and his team skedaddled out of town.

About one month later in the town of Essonnes, after establishing a new first triangle, Delambre had a nearly identical encounter in the tower of the Chateau de Belle-Assise. However, this time it was the National Guard who after being notified by local authorities came to arrest Delambre.

A long interrogation ensued and just as before, the gathering crowd was highly skeptical of Delambre's explanation of the purpose of his measurements and the use of his equipment. The National Guard ripped through the expedition's belongings discovering the royal passports as well as a series of envelopes with the royal seal. As with the previous encounter, Delambre pursuaded the arresting authorities to take the matter up with the local authorities. Once again, Delambre and his team were escorted to the town square.

Another round of long interrogations ensued. Then the matter of the letters arose. Delambre had mixed feelings about the letters. As with the passports they had the royal seal, but maybe they would corroborate Delambre's responses throughout the interrogations. The interrogators unsealed two envelopes and read the letters. They indeed corroborated Delambre's statement that he was on a scientific mission supported by the Academy. The crowd insisted upon unsealing more envelopes. At this point Delambre put his life on the line.

Delambre suggested that the interrogators randomly select one envelope and read its contents. If the letter was not identical to the preceding two, they could execute him. Otherwise there would be no more opening of the letters. While Delambre was watching his interrogators unseal yet another letter did time stop? While the Earth stopped rotating did Delambre imagine a mix-up whereby the envelope contained the wrong letter that was meant for another purpose. Perhaps the letter contained orders for an Army commander. But the Earth kept spinning and time resumed. The interrogators confirmed that this letter was identical to the preceding two.

Delambre's level headed explanations along with his life-risking proposal persuaded the authorities to suspend their suspicions and look further into the matter. The authorities sent an official to Paris along with a copy of the letter. Within several days the official returned and confirmed that the expedition was on a scientific mission sponsored by the Academy; and so the local authorities released the expedition from their imprisonment. Delambre did not press the issue of completing his observations, but took the opportunity to hastily depart.

Close Call in the South

Contrary to Delambre's complete lack of progress in the north, the Machine was on a tear in the south. Mechain arrived in Barcelona in early September, 1792. The director of the Barcelona observatory, Jose Esteve, enthusiastically welcomed Mechain. King Carlos' government was less enthusiastic about the presence of a representative of a republican government that gained power by overthrowing the French King. As if this wasn't bad enough, the ex-King, Lois XVI happened to be King Carlos' cousin. Nevertheless, the authorities accepted Esteve's explanation that the purpose of Mechain's mission was scientific and after some time authorized the survey to proceed. Esteve of course, along with two to three miltary personnel would keep an eye on Mechain and assure that he was up to no shenanigans.

By the time Mechain received authorization to proceed, it was autumn and snow was visible on the peaks of the Pyrenees. Mechain had no previous experience hiking along mountain ranges as serious as the Pyrenees, let alone winter mountaineering. The pleadings from Esteve along with warnings of the dangers could not dissuade Mechain from immediately getting on with his mission. Instead of convincing Mechain to prudently await for spring, Mechain obliged Esteve to accompany him along a perilous path.

Esteve accompanied Mechain and his small retinue up many peaks but some were beyond his reach. The slopes were too steep, the cliffs impassable, the snow too deep, the danger was in his face. During those times, below in a safe place, Esteve might worry and pray for Mechain's safe return. The Machine returned every time, successfully marking his triangles and taking his observations. Esteve must have been awe inspired. But there was one most annoying law that Mechain laid down. He would not permit Esteve to learn how to use the repeating circle and not allow him to take a single observation. Esteve could not so much as lay his hands on the device. He could only observe Mechain measuring the angles.

By January 1793, Mechain surveyed nearly all of the Spanish territory within his responsibility. He had completed far more than he originally set out to do. He contentedly decided to spend the remainder of the winter in Barcelona where he would take observations of the stars so that he could determine the latitude of Barcelona. Then in the spring he planned to return to the Pyrenees, complete his triangulation within Spain, pass on to France, and continue northward. But the stars had a different plan.

Upon retreating from the mountains to the hospitable seaside city of Barcelona, Mechain accepted Esteve's invitation to settle in his observatory upon Montjuic, a hill that was once a cemetery site for the pre-inquisition Jewish community. Subsequently the hilltop hosted a military fortress and within the fortress was the observatory. At first the authorities balked at allowing Mechain to take up residence aside a military encampment. But Esteve used his influence and once again cleared the way. Esteve even assisted Mechain in Mechain's construction of observation platforms for Mechain's exclusive use.

Given the hospitality that Esteve showered upon Mechain, Esteve would have most certainly agreed to a collaborative effort toward improving the estimate of Barcelona's latitude. But the Machine was not designed for collaboration. He was singular in his pursuit. He alone would take the observations, record them, analyze them, and from the observations, compute the latitude. This was a design flaw that would later be instrumental in the Machine's breakdown.

Despite King Carlos' efforts to avoid war, the French had other ideas. On January 23, 1793, the republican government of France severed its ties to the past by severing Louis XVI's head at the guillotine. This was a direct insult to King Carlos, Louis' cousin. On March 7 France went beyond insults and declared war upon Spain.

The war interrupted Mechain's observations from Montjuic; it was untenable to allow this Frenchman to remain at the military site. Esteve arranged an alternative residence at the Fontana de Oro hotel. The Spaniards would not allow his departure from Spain until the two nations were at peace. But until that time, they would be very hospitable. Mechain was both a hostage and a respected guest.

Instead of finalizing the triangulation in Spain and moving on to France, Mechain was stuck for who knows how long in political quicksand. To their credit, both Mechain and his hosts made the best of it. As part of making the best of it, the most esteemed members of the scientific community of Barcelona extended their greetings and invitations to special occasions.

One such individual was Barcelona's preeminent physician, Doctor Salva. The scientific community of Barcelona was most proud of a hydraulic pumping station erected by their engineers. Dr. Salva extended an invitation for Mechain to inspect the facility. During their visit, an accident occurred that endangered both Dr. Salva and his assistant. While Mechain was stingy and uncooperative in his work, he proved to be most courageous and giving when the moment mattered. Mechain rushed to assist the two endangered men. In doing so, he put himself at risk and the dice rolled its decision. A beam struck Mechain rendering him unconscious with broken ribs, a broken collar bone, an arm injury (possibly a torn tendon), and an injured chest cavity.

Doctor Salva transported Mechain to a nearby home where he attended to the man who was obviously on the precipice of death.

Recovery in the North, and Then..

For Delambre the year 1792 was an account of drama with little progress. At the outset of 1793, the beheading of King Louis XVI presaged more drama and as if on queue, Delambre's drama continued. The previous year's harassment from local authorities became routine. When the people weren't thwarting Delambre, the weather went on the attack. Rain, drizzle and accompanying fog obscured the surveyors' views. Furthermore, August 1793 saw the abolishment of the French Academy of Sciences. And so 1793 ended worse off than 1792. Little had been accomplished and the projects most ardent backer was no longer.

After some consideration Robespierre's government viewed the survey with favor and placed the project under the control of alternative government agencies. But its position was tenuous.

The spring of 1794 began a new surveying season. With the change in season, Delambre decided to make a change as well. Rather than starting from Paris, he would proceed to Dunkirk and work his way south to Paris and beyond. It looked like an eternal curse had befallen Delambre. On his way to Dunkirk, news of a successful Prussian assault circulated, it would not be long before Dunkirk would fall.

Dunkirk did fall, but Delambre outpaced the Prussians. Luck switched sides. The weather was cooperative and Delambre took advantage. He made his way from Dunkirk to Paris in record speed. More luck, in July the arrest of Robespierre ended the reign of terror. After one final arrest and release of his team, Delambre proceeded without harassment. This year Delambre the leader showed what he was made of. He along with his team completed one third of their survey, triangulating well south of Paris to the city of Nemours.

And just like that, luck switched sides again. The post- Robespierre government soured on the project. After three years effort, the survey was two years behind schedule and was at best one third the way complete. At this pace it would take another six years. And all for what?

The Academy had been anticipating the government's impatience and decided to put forward a provisional length for the "meter" before its dissolution. Using available data, the preeminent mathematician Pierre Laplace declared the provisional meter to be 0.513426 toises. For those unfamiliar with the toise, the value was 3 pieds 11.44 lignes. This confusion provides the perfect example for why the standard "meter" was necessary.

Alongside the calculation of the provisional meter was a refinement of the flattening coefficient. Recall Newton's estimate was 1/230. Clairaut, using Jacques Cassini's measurements along with measurements from the Lapland expedition revised the estimate to a dumpy 1/170. And now, using measurements from both the Lapland expedition and the Peruvian expedition, Laplace weighed in at 1/320. Hooray, Earth was getting more fit. Laplace announced his findings in November 1793 when the Academy was no longer. In the same month Robespierre's government arrested Lavoisier who was on the same side of the government's arguments against the survey. But as a tax collector for Louis XVI was on the wrong side of the politics of the day.

Putting the provisional "meter" into play had its own risks. Echoes of Lavoisier surfaced. The whole survey was based upon a false premise of perfection. Can anyone guarantee that the new survey will add anymore accuracy to the estimate or that the estimate will not be further revised at at later date? Why not accept the provisional "meter" as the final "meter" and move on? Although the echoes of Lavoisier were in the air, they did not come directly from Lavoisier's breath. In the final months of the reign of terror, Lavoisier followed Louis XVI to his headless fate. Nevertheless within the government, echoes of Lavoisier did find many supporters.

And just like that, luck switched sides again; the gig was up. Support from the government was not forthcoming. Delambre received orders to halt; no more triangles. As the project ended, the year 1794 also ended.

In 1795, following the echoes of Lavoisier, the government prepared to make the provisional meter the permanent meter. Production of meter sticks commenced. Distribution of the sticks was meant to educate the country and provide officials with the means to enforce the fair measurement of products used in commerce by a common standard. The result was a disaster. One could blame the insufficient production and distribution of meter sticks. One could blame the lack of educational material. While true, the real culprit was a deaf government. Nobody, save a few members of the now defunct Academy wanted the damn meter. Everyone understood their local standards of measurement and had confidence in the fair administration of those standards by their local authorities.

And just like that, luck switched sides again. On October 25, 1795, the government created a new agency, Institut National des Sciences et des Art. Within this agency, the government resurrected the Academy of Sciences. The old team of colleagues was back together with open support of the government. The disastrous roll out of the meter proved to be a gift to the majority of the Academy who wished to continue the survey. They pressed their argument and by the beginning of 1796, the survey was back on and not to be switched off until completion.

For Delambre, a stable political climate assured that luck was no longer part of the equation. His survey would rest upon his skill. Delambre proved he was up to the task. In the year 1797 Delambre's team completed the triangulation of their portion of the survey. Alongside the survey at the request of the Academy, Delambre's team performed three additional latitude measurements. Then in 1798, they topped it off with the measurement of the northern baseline near Malun. And for good measure, Delambre oversaw the measurement of the southern baseline at Vernet.

Broken

What was Dr. Salva thinking as death hovered over his unconscious patient? Salva was guilt ridden and attended to Mechain daily. But medical knowledge of the day was limited. Salva knew there was little he could do but watch. And to his surprised he watched Mechain recover. Within two months, the Machine was on his feet. Later Mechain remarked, "Because of Doctor Salva I nearly died. If it weren't for Doctor Salva I would have died."

In some ways Mechain was well suited to the life of a hostage. His feet might be bound to Spain, but his eyes were open to the universe. Mechain pursued his passion, astronomy. Mechain set up an observatory on the deck outside his room at the Fontana de Oro. The dizzying pace of observations that made him famous in Paris, now continued in Barcelona.

Then in 1794, it happened. This perfect Machine discovered an error in his observations. The latitude measurement that Mechain performed from Montjuic was incommensurate with that from the deck at Fontana de Oro. It was off by a measure of three seconds that Mechain frantically attempted to account for. But there was no accounting for the error. The Machine knew that he screwed up.

Mechain single handedly mapped more of the skies than any of his counterparts throughout Europe. He conquered the Pyrenees. He fought off death from a blow that would have killed any ordinary man. But this three second error that exposed his imperfection broke him. He never recovered.

What is worse, living with the knowledge that you are a fraud, or coming clean and exposing your flaw to your peers? Mechain chose the former. Only he had taken the observations, only he knew of the error, only he controlled the log book, and only would know his secret. This choice extracted its price.

At one point during his survey, Delambre also had conflicting observations. He had a path for addressing the resulting errors that Mechain could not take advantage of, collaborators collaborating. Working with his team members, Delambre discovered that the rotating circle had a broken component. He returned the rotating circle to Paris for repairs and then continued.

Without corresponding measurements from collaborators, and unwilling to discuss the matter with others, Mechain never discovered the likely source of the error. The issue was not Mechain's observational skills. A subsequent analysis proposed that a component within the rotating circle had warn down, causing a slight misalignment. Ironically, it was Mechain's observational skills that most likely caused the piece to wear down; his proliferate observations were more than the poor rotating circle could handle.

The heavens no longer provided refuge to Mechain. He wished to continue the survey, but the ongoing war held him hostage. Then quite remarkably, Esteve once again came to Mechain's aid. Through a bureaucratic blunder, the Spanish authorities granted permission for the survey to continue within Spain under the supervision of Spanish escorts including Esteve.

It bedazzles us that Mechain had dispatched his most capable assistant, Jean Tranchot to penetrate the boarder and triangulate into France without awakening any alarm bells in the heads of the escorts. The alarm bell's clapper should have punched holes in the brains of the Spaniards when, for the only time throughout the entire expedition, the rotating circle was not securely in the hands of Mechain.

For once, and never again, Tranchot made measurements with the rotating circle. It is almost a certainty that he took advantage of the situation and passed his survey results onto the French where they fell into the hands of the French Army. Throughout this entire episode, Esteve vouched for Mechain.

Meanwhile, the war effort see-sawed. In 1793, France may have regretted declaring war upon Spain as the Spanish Army under General Anotonio Ricardos won battles on French soil. After the death of General Ricardos (March 6, 1794), the French under General Dugommier kicked the Spaniards out of France and pursued them into Spain. On November 18, 1794, the Spaniards took their revenge on Dugommier killing him in battle. Three days later, the French returned the favor by killing the Spanish general, Conde de la Union.

In 1795, the Spanish began serious diplomatic efforts to end the war. It was during this time of turmoil that Mechain made his move. In May, he secretly arranged for passage to Italy by sea and disappeared. In June, he arrived in Genoa. On July 22. 1795, the signing of the Treaty of Basel in which Spain recognized French sovereignty over the areas they controlled, formally ended the war. Mechain's personal war was just heating up.

The war had interrupted Mechain's communication with the Academy. Mechain was unaware of the political events that occurred during his absence. The Academy was unaware of Mechain's whereabouts let alone the status of his survey. On July 11 Mechain presented himself to the French consulate in Genoa where he sent three letters to the Commission of Weights and Standards, Delambre, and his wife.

The Commission was a political arm that oversaw the Academy's work. Leading scientists were members of both the Commission and the Academy. The practical effect of writing to one was the same as writing to the other. The political effect of writing to the Commission was to demonstrate that Mechain's allegiance was to the government. In the letter Mechain apologized for the delays in the survey, explained the difficult circumstances that he confronted, requested funding to continue with the survey, and requested instructions for how to proceed.

Delambre received another over the top apology that was perplexing; it seemed totally unnecessary given the wartime conditions. Madame Mechain read that her husband was looking forward to their quick reunion.

The Academy members close to Mechain, Delambre and Legendre, who happened to also be on the Commission formally replied on behalf of the Commission. With compassion, they assured Mechain that they fully understood the difficult circumstances which hindered Mechain's progress. They requested that he forward his log book. And they also requested that Mechain return to Paris so that they could receive a first hand account of the status of the southern survey and perform an analysis of the data. These were genuine requests made to a colleague whose contributions they valued.

Once again, Mechain confronted two choices. Return to Paris for the long awaited reunion with his wife and expose his shameful secret to his colleagues. Or, continue his isolation from colleagues and family so that he could safeguard his secret. Mechain chose the latter. He did not return to Paris and kept his log book to himself. For the remainder of 1795, Mechain holed up in Italy.

In 1796, Mechain left Genoa for Marseille. New place, same old behavior; Mechain holed up in Marseille. Another year passed without progress. In 1797, Paris granted funding and support for Mechain to continue his field work. Mechain's body had recovered; he was physically as able as he was before his accident. A fully committed Mechain could have finished the work in 1797. But while Mechain's body was restored, his spirit was broken.

It was obvious to all that Mechain was not the Machine that his colleagues remembered. They did not know the source of his despair, but it was on full display. Madame Mechain also grasped that her husband suffered from anxiety. She decided to travel to the south and comfort him. After six long years, the reunion ended in frustration. Mechain refused his wife's words of support. He refused her promise to finish the survey by his side. He refused her pleadings to stop his self-imposed torment. Madame Mechain's return journey to Paris must have been heart wrenching.

As for the survey, 1797 passed and although there was progress, Mechain's triangles were a considerable distance from Delambre's.

The 1798 surveying season commenced with urgency. After several victories, France was the dominant European power. The government sought to increase its prestige by announcing their metric standard. The Academy organized a convention inviting prominent European mathematicians to attend. The convention would be an open scientific investigation of the methods and data collected by Mechain and Delambre. Each scientist would have access to the data and be able to draw their own conclusions. Scientists began to assemble in July.

There was a glitch, Mechain hadn't yet finished his triangles and refused to get a move on it. The prestige of France and the Academy in particular was at stake. Only after the Academy desperately bribed Mechain with the position of Director of the Paris Observatory upon his return did Mechain complete his triangles. The mood among the scientists who had been kept in wait for nearly five months was quite foul upon Mechain's November return. It didn't get any better.

Mechain refused to hand over his data. He kept the world in wait without explanation. Whatever pressure was applied, it didn't work. Until in January of 1799, Mechain gave in. He released his data and subjected himself to the scrutiny of his peers. It went swimmingly well. All those years of worry for his reputation and in the end, his reputation soared even higher.

The calculations started. Laplace examined the latitude data and was disturbed to find that the flattening coefficient overreacted. It went beyond its previous record all the way up to a dumpy 1/150. Laplace incorporated additional data and Earth became more fit than ever with a flattening coefficient of 1/334.

By means of various publications, the data circulated throughout Europe. In Gottingen, Friedrich Gauss (1777-1855) made his own calculation based upon the four measurements of Delambre and Mechain alone. Gauss' verdict, dumpy at 1/187.

But the moment all had been waiting for arrived. On June 22, 1799, the meter got a demotion from its provisional parent by 0.144 lignes. It came in at 3 pieds, 11.296 lignes or equivalently for those more familiar with the toise, 0.513074 toises. By the meter's own standard, the difference between the provisional meter and its official update is 0.32 millimeters, about as thick as three sheets of paper.

How close was the measurement by today's standard? The best measurement of the average distance from the equator to the north pole is 10,000,196 meters. Using today's standard, the meter is short by 0.196 millimeters, around the thickness of two sheets of paper. This is exceptional, we can only marvel at the level of precision in the face of the challenges. But wait, let's consider the provisional meter. It is long by 0.128 millimeters. The provisional meter was more accurate than the actual meter. The years of struggle, the near death experience, the arrests, the captivity, the turmoil, the quest for exactitude and perfection, it all ended up just as Lavoisier had predicted. Exactitude and perfection don't exist, but we can determine what works.

Among the remains in the Catacombs of Paris are those of Antoine Lavoisier. You can visit the Catacombs at Avenue du Colonel Henri Rol-Tanguy. As you make your way through, listen for the faint sound of laughter. If there is any justice it is the eternal laugh of Lavoisier.

The Aftermath

The advantages of measurement standards were evident enough for the meter to become popular in Europe. There was a notable exception, France. Just as the people rejected the provisional meter, they rejected the meter. Did anyone have a realistic expectation that changing the value by an imperceptible 0.144 lignes would change the public's mind?

As industrialization and trade increased, standardization became even more critical. In 1875 France hosted an international meter convention. Scientists weighed in on the the adoption of the 1799 meter. Revisions of the meridional length indicated that the meter came up short, but by that time an update was infeasible and the convention declared the meter sacred. The meter spread beyond Europe to all continents. Even the French people capitulated. There is one notable country that is still a hold-out, the U.S.A..

The definition of the meter has evolved in accordance with the scientists' desire to tag its length with an eternal constant of nature. In 1960 the official definition was 1,650,763.73 wavelengths of the orange-red spectral line of krypton-86 in a vacuum. In 2002 scientists improved upon this definition and it remains today at the distance that light in a vacuum travels in 1/299,792,458 seconds.

What happened to Mechain and Delambre? Mechain never moved on; the error consumed him. At age 60 he insisted on leading an expedition that would extend the meridian measurement southward from Barcelona to a chain of islands in the Mediterranean. Perhaps he hoped to find restitution with the successful conclusion of this survey. A mosquito put an end to his plight. In 1804 on one of the offshore islands he died of malaria. Delambre recovered Mechain's original logs that while he was alive, Mechain kept private. To his horror Delambre discovered that Mechain fabricated the data which he presented to the 1799 commission. That data which anchored the meter was a lie. The meter was a lie.

6.5 Parsing the Data

The principle behind the success of the repeating circle is: averaging over many observations can improve the accuracy over a single observation. Provided there is no error bias, errors cancel out. This principle was in fact used to accurately determine the period of planets' orbits around the sun. The Chinese and ancient Greeks determined the Earth year quite accurately by averaging the days between equinoxes (spring or fall) over many years.

It is natural to want to extend this concept in the following sense. Suppose that one has a lot of data, more than is necessary to determine parameters of interest. We wish to use all the data to estimate the parameter with great accuracy. Furthermore, computing the estimate should not be overly burdensome. In 1805, the mathematician Adrien Legendre published such a method. He applied it to estimate the orbit of comets using an abundance of data.

Legendre received the praise that he deserved. Praise came from Gauss at Gottingen, but this praise had a different twist. Gauss praised himself, for he claimed that he discovered the method first. And then started a public fight.

In truth, Gauss' claim is not without merit. The view among the scientific community is that Gauss was the most capable European mathematician of his era. But if you don't publish it, you can't deny others their rightful claim. And you certainly shouldn't be starting a public campaign to smear the man who clearly independently discovered the method.

One hint that Gauss did discover the method prior to Legendre comes from a 1799 letter to the publication *Allgemeine Geographische Ephemeriden* where Gauss presents his analysis of Mechain and Delambre's meridian data. There, Gauss cryptically refers to "meine Methode" and presents his result that the flattening coefficient is 1/150 without presenting the computations that yield the result. In a subsequent letter, Gauss notes an error in the published data and adjusts the flattening coefficient to 1/187; once again there is no trace of an explanation for "meine Methode".

There have been failed efforts to replicate Gauss' result using the least squares method. From these efforts, one cannot conclude that "meine Method" was not the least squares method. Application of the method requires an estimation of arc lengths along the meridian between specified latitudes. We do not know what approximation Gauss used, and perhaps this is the source of different outcomes.

Below, we follow in the steps of previous mathematical detectives and apply the least squares method to the Mechain-Delambre data.

The Method and AI

Table 5.2 along with the following explanation is directly from Stigler's 1980 article published in *The Annals of Staistics*. Note that in the explanation below, Stigler converts the distance measurement, "modules" to feet. *The Annals of Statistics* is a product of USA.

Segment	Modules S	Degrees d	Midpoint L
Dunkirk to Pantheon	62472.59	2.18910	49° 56' 30"
Pantheon to Evaux	76545.74*	2.66868	47° 30' 46"
Evaux to Carcassone	84424.55	2.96336	44° 41' 48"
Carcassone to Barcelona	52749.48	1.85266	42° 17' 20"
Totals	275792.36	9.67380	

Table 6.1: French arc measurements, from *Allgemeine Geographische Ephemeriden*

The number 76545.74 is a misprint; the correct number is 76145.74. The table gives the length of four consecutive segments of the meridian arc through Paris, both in modules S(one module ≈ 12.78 feet) and degrees d of latitude (determined by astronomical observation). The latitude of the midpoint L of each arc segment is also given.

Gauss' initial letter uses the data with the misprint. The subsequent letter uses the correct number, 76145.74. Below, we use the corrected data. It should be noted that the degrees d, gives the latitude difference between the endpoints of the segments.

A commonly used model to determine segment length along a meridian is the following.

$S = d(S_e + y\sin^2 L)$ where

- S is the length of a segment lying upon a meridian (same as Table 5.2).
- d is the degrees covered by the segment (same as in Table 5.2)
- S_e is the length of a one degree segment along a meridian at the equator.
- y is the difference in length between S_e and a one degree segment passing through the pole.
- L is the midpoint latitude of the segment (same as Table 5.2).
- The equation assumes that the shape of the Earth is a spheroid. The shape of a meridian is then an ellipse with its longer axis along the equator and shorter axis between the poles.

Comparing the formula for a segment length with the data from Table 5.2, the formula contains two unknowns, S_e and y. Using two entries from the table we could create two equations and solve for the two unknowns. For example, using the first two entries gives the following equations.

$$62472.59 - (2.18910)S_e - (2.18910)(0.585821)y = 0$$
$$76145.74 - (2.66868)S_e - (2.66868)(0.543800)y = 0$$

From these two equations values of S_e and y are obtainable. But hold on, suppose we choose another two sets of equations, will we then get the exact same values for S_e and y? The conditions for getting the exact same values are:

- The formula is perfect.
- The Earth is a perfect spheroid.
- The data is perfect,
- The computation of the squared sine of the angles is perfect.

None of these conditions hold. The value of S_e and y are not the same across all pairs of equations. Instead of selecting a single pair, shouldn't we approximate S_e and y using all of the available data? Yes. We cannot possibly find values for S_e and y that permit the formula for the length of the arc segment to be satisfied for each observation, but we can find one that is an excellent compromise. That is, it may not be perfect for any of them, but it's pretty good for all of them. This is the approach of Legendre and Gauss.

Instead of setting the right hand side of our equations to zero, let's include every equation and set it to an error, ϵ_i. There are four equations with four errors, i ranges from 1 to 4.

$$62472.59 - (2.18910)S_e - (2.18910)(0.585821)y = \epsilon_1$$
$$76145.74 - (2.66868)S_e - (2.66868)(0.543800)y = \epsilon_2$$
$$84424.55 - (2.96336)S_e - (2.96336)(0.494706)y = \epsilon_3$$
$$52749.48 - (1.85266)S_e - (1.85266)(0.452753)y = \epsilon_4$$

Summing across all squared errors informs one of how large the error is across all entries. What Legendre and Gauss did was to figure out how to select S_e and y so as to minimize the expression:

$$\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2$$

Because every entry in Table 5.2 gets equal say in the above expression, the minimization of the sum of square errors turns out to be a very good compromise. If the error entries follow certain properties, then one can make the error as small as desired by increasing the number of observations[7].

Finding the minimum looks like a formidable problem. You have to square each of the four expressions for the four errors, add them together and then somehow find the best values of S_e and y. It's not as bad as it looks. Using Newton's calculus, solvable expressions for S_e and y emerge[8]. Another equation allows one to determine the flattening coefficient from the values of S_e and y.

For the reader interested in the details of these computations, there are many sources available. In this book, our aim is to explain the significance of what Gauss and Legendre achieved.

Above there are only four entries in the table with only two unknown values. In the days of Legendre and Gauss, cranking out an answer by hand is a good day's work. But what have they done for us in our age? Today we can create tables with literally billions of entries and use it to fit models with billions of unknown parameters; the dataset used to train ChatGPT is indescribably large and the underlying model has billions of parameters. Legendre and Gauss provide the first available method for selecting the best unknown parameters using the billions of entries in the table. Based on their method, we can program a computer to find very good solutions for the unknowns. This is the secret behind today's AI.

6.6 Gauss, the Data Scientist

We shall never know whether Gauss took his "Methode" to the grave or if he used the least squares method to analyze the Dunkirk–Barcelona meridian data. Here's what we do know.

- If Gauss' "Methode" was not the least squares method he abandoned it in favor of the least squared method.
- Legendre published the first description and application of the least squares method. Gauss' subsequent publications furthered the theory by applying probabilistic methods to analyze the error. Gauss proved that under favorable conditions, the error approaches zero as more data is used[9].
- Both Legendre and Gauss applied the least squares method to determine the orbit of comets or asteroids. Gauss also applied the least squares method to interpret surveying results that he performed in the vicinity of Hamburg.

This section presents Gauss' calculation of the flattening coefficient from the perspective of the program of the modern data scientist as described in Chapter 2. There is one missing component, validation. As this chapter's description of the expeditions notes, data collection for this endeavor was excessively expensive and hard to come by. Gauss used the available data to make a calculation, validation was not his aim, nor did he make use of the results in any future works.

[7] The expression represents the square of the length of the error vector in four dimensional Euclidean space. Minimization yields the error vector with the shortest length in four dimensions. Increasing the number of observations decreases the length of the error vector

[8] There is also a geometric approach that does not require calculus. This approach generalizes the notion of distance between two points to arbitrary dimensions and then uses the Pythagorean theorem in higher dimensions to find minimal projections onto a linear subspace.

[9] Technical conditions from probability theory are necessary to apply this statement. The process producing the data must be unbiased and the standard deviation of the process must exist.

It should be noted that Gauss expanded his method of least squares to make best estimates of angles and distances from survey data. There are inherent errors in survey measurements. Using least squares methods, Gauss applied corrections that minimized errors. With these corrections, Gauss was able to establish longitude and latitude of survey points very accurately. Gauss did collect additional survey data to validate his results.

The analytics of Gauss' survey methods are beyond the scope of this book. Below, we stick to Gauss' 1799 calculation of the flattening coefficient.

Define the problem.

Determine the flattening coefficient of the Earth.

Propose an input-output parametric model of the system.

Using spheroidal geometry, calculus, and approximation methods, determine the length of a meridional arc between two latitudes. An example of such an approximation is:

$S = d(S_e + y \sin^2 L)$ where

- S is the length of a segment lying upon a meridian (same as Table 5.2).
- d is the degrees covered by the segment (same as in Table 5.2)
- S_e is the length of a one degree segment along a meridian at the equator.
- y is the difference in length between S_e and a one degree segment passing through the pole.
- L is the midpoint latitude of the segment (same as Table 5.2).
- The equation assumes that the shape of the Earth is a spheroid. The shape of a meridian is then an ellipse with its longer axis along the equator and shorter axis between the poles.

As noted in the preceding section, Gauss most likely developed his own approximation. The parameters of this model are S_e and y. The output is the arc length, which is both calculated as above and then compared against data.

Identify the required data.

The dataset should have measured arc lengths across many meridional arcs. In practice difficulty in obtaining such data limits the number of observations.

Collect and organize data as inputs and outputs.

The inputs into the model are:

- d the degrees in latitude covered by each arc.
- L The latitude of the midpoint of each arc.

The outputs are S, arc lengths of each arc. These are data points that one compares with the modeled output.

The data may be arranged as in Table 5.2

Define a metric that quantifies the error between model predictions and observed outputs.

Gauss used "Meine Methode" which may have been the least squares method. For the least squares method, the sum of the square errors between the data and the model's output gives the metric.

Apply an optimization routine to adjust the parameters and minimize the error.

Assuming a least squares method, using linear algebra, Gauss developed solutions for the values of S_e and y that minimize the difference sum of square errors.

Validate results against additional data.

There was insufficient data for cross validation.

6.7 Final Thoughts

In the earlier days of Aristarchus and Ptolemy, data analytics was primarily confined to determining the parameters of a model by exact fitting. Using this approach, one obtained sufficient data to determine the parameters of interest.

The Chinese astronomers under Guo Shoujing as well as Kepler acquired and used more data than was necessary to fit their models. However, when fitting their models; they used the same process as Aristarchus and Ptolemy. From their large dataset, they selected the required number of observations to precisely determine model parameters. Then they would cross-check the results against other observations and make adjustments. It was an ad-hoc approach toward matching model results to many observations.

The least squares method of Legendre and Gauss was the first systematic approach toward fitting model parameters to an abundance of data. This was a major advancement in data science; this method is central to ChatGPT's success.

Success breeds success. The next chapter describes Francis Galton's approach to not only using data to fit models, but also to extract information and describe relationships between different phenomenon.

6.8 Summary Poem: The Perfect Fit

From Newton's desk, the Earth was round,
Bound by laws where truth is found.
He saw through stars and falling fruit,
And shaped the Earth to match the root.

Not sphere, but squashed at polar ends,
Where gravity and spin contend.
The equator bulged, the poles were pressed—
An oblate spheroid was his best guess.

Then France, with maps and monarch's pride,
Set Cassini's clan on journeys wide.
But data clashed—was Newton wrong?
Did Earth run lean instead of strong?

To Lapland's cold and Andes' flame,

Brave minds and bodies staked their claim.
With sextants packed and flares in hand,
They tramped through snow and burning sand.

Maupertuis proved Newton right,
While La Condamine, in jungle night,
Fought fevers, frost, and bitter foes,
To trace the planet's swelling nose.

Then came the meter, grand and bold—
A length from Earth, from pole to fold.
Delambre led through fear and storm,
While Mechain broke from his own norm.

A broken soul with comet's sight,
He hid a flaw he couldn't right.
A man who braved the Pyrenees,
But fell before three arc-seconds' tease.

And though perfection slipped away,
The measure held, and still holds sway.
From Newton's thought to AI's birth,
A world revealed: the shape of Earth.

No more by guess or line of sight—
Gauss made the errors yield to right.
With least squares penned by steady hand,
He let the data make its stand.

From comet arcs to geodetic scans,
He built a bridge with bold new plans.
Fit truth through noise, he dared to dream,
A model born from scatter's scheme.

And now in code, his wisdom wakes—
In AI's gears and learning lakes.
From fitting lines to neural turns,
The fire of least squares still burns.