# Chapter 5

# Kepler's Wars: Mars on Earth

There are moments in history where time seems frozen. Yesterday's truths are the same as today's and appear to hold for eternity. Then there are moments in history where yesterday's truths shatter undeniably right in front of humanity's eyes and new constructs arise.

Kepler was caught in religious and political upheavals that resulted in deadly wars and changed the political structure of Europe. He participated in a scientific revolution that changed the very essence of how humanity saw its position in the universe. One admires the persistence and brilliance with which Kepler successfully reveals the heliocentric orbits of the planets. Admiration adjoins empathy when presented with the harsh environment that confronted Kepler as he discovered the pathway of the heavens. Kepler approached his scientific efforts with an extreme passion. Perhaps it was this passion that allowed him to maintain sanity while surrounded by nonsense. This chapter explores Kepler's battles with both Mars, the difficult circumstances, and Kepler's passion.

## 5.1 Unbelievable

Imagine owning your own island; a peaceful refuge far away from the turmoil engulfing those on the mainland. Imagine living in luxury that is reserved for only a handful of men among millions. Imagine having your nose chopped off because in your youth, you got into a heated argument with a fellow student over an equation. The argument escalated to the point where the combatants drew swords and your nose was the victim. No problem, you can afford a prosthetic replacement of solid gold. Imagine having unlimited support for your pet hobby, astronomy. The support includes the finest equipment for the purpose of monitoring the heavens that is available to no one else on Earth. Imagine being able to staff your observatory with the most devoted cohorts who share your passion for understanding what's up there. Imagine being able to provide full economic security for your cohorts so that they don't have to worry about what's going on down here. Imagine having exotic pets, like a moose who along with your cohorts is a guest at your feasts and gleefully participates in the consumption of unlimited quantities of beer.

This sounds like it would make a nice movie, fiction with one flaw. Nobody would believe it. Mark Twain once remarked that nonfiction is always more impressive than fiction, because fiction has to be believable. We may continue with the story because this is nonfiction. It is the life that Tycho Brahe (1546–1601) enjoyed and our continuation is equally unbelievable as the preceding paragraph.

Tycho Brahe's fortune rested upon his Uncle Jorgen's misfortune. In a heroic effort to save a drowning man, the nobleman Jorgen Brahe dove into the icy waters of the North Sea and rescued the man from certain death.

The downside for Jorgen is that he he became ill and succumbed a few days later. The upside for Tycho is that the saved man was none other than King Frederick of Denmark.

Jorgen Brahe was childless and adopted his nephew, Tycho as his own. As an offering of gratitude and in commemoration of Jorgen, King Frederick presented opportunities to Tycho that few others enjoyed. Tycho proved worthy of the opportunities Frederick gifted; he became a respected member of King Frederick's court. He apparently had a knack for inner-court politics and was able to offer prized advice. In return for his services, King Frederick bestowed upon Tycho the island of Hven along with the economic means described above.

Life upon Hven was idyllic, but there were hiccups. Tycho spoiled his pet moose with generous servings of beer. One day, the moose imbibed a little more than he could handle. Irresponsible drinking has its consequences. At least the moose died happy. Perhaps in his drunk stupor he enjoyed tumbling down the stairs prior to breaking his neck at the bottom. The incident portended Tycho's own downfall; which we will come to later on.

Meanwhile as Tycho was enjoying his lavish Hven lifestyle, a commoner Johannes Kepler (1571 – 1630) struggled with harsh realities. Kepler's father was a mercenary who abused his wife and son. The job required frequent travel which brought a spell of relief into the household. The job also came with its own hazards which eventually caught up with the elder Kepler. The death of Johannes' father brought about a permanent cease of the beatings, but left the family impoverished.

The Lutheran community at times would offer a scholarship to commoners with promise. This afforded a tiny minority of the majority with the opportunity of an education that was nearly exclusively reserved for the upper-crust. Through luck or well administered policy or a combination of both, Lutheran authorities became aware of the unusually gifted Johannes Kepler and offered to fund his schooling up through the Lutheran ran university in Tubingen.

The disdain from his fellow classmates of noble birth resulted in bullying of the commoner, but hey it was a step up from the abuse that his own family dished out. There was genuine intellectual stimulation that Kepler reveled in. In the early years, teachers who appreciated Kepler's intellectual curiosity protected him. Later his intellectual precocity and commitment to intellectual honesty caused a schism between the Lutheran educators and Kepler. Intellectual honesty was a trait that would lead to a lifetime with bouts of danger and poverty, but was also central to his discoveries for which he is a historical figure. We will discuss these discoveries later in the chapter.

Kepler with little social standing and no family support turned to God as the central stabilizer in his life. He desperately wanted to serve God as a preacher. His teachers had other ideas. Kepler liked to think things out for himself and with his intellect, could out argue his teachers. In their frustration with his disputes over their authority, they marked him as socially unreliable. Better not give him a congregation. Who knows what he might say?

Unfair as this might seem, we are fortunate that the educators at Tubingen recognized Kepler's prodigious mathematical talent and nudged him in the direction of mathematics teacher. The disagreeable Kepler was surprisingly agreeable. Mathematics is seductive. One deduces absolute, indisputable truths. Through mathematics, Kepler could discover God's truths.

Moving through the years, in 1596, while working as a mathematics and astronomy teacher at a Lutheran school in Graz, Kepler published his first work, *Mysterium Cosmographicum*. Written in Latin, the intellectual language of the day, *Mysterium Cosmographicum* establishes a relation between the planets' positions in the heavens and the Platonic solids. Kepler proposed that the distances between the six known planets (at the time: Mercury, Venus, Earth, Mars, Jupiter, Saturn) could be explained by nesting the five Platonic solids between their orbits, each enclosed in a sphere.

From his position in Graz, located not too far from Vienna, Kepler was convinced that through the beauty of mathematics, he captured the essence of God's design. He certainly captured the imagination of astronomers about Europe. Of course we now know that this is entirely fiction, but following Mark Twain's dictum, at the time it was believable. What was not fiction was the mathematical skill required to embed the Platonic solids within the planets' orbits. A genius announced himself to the world. Let's go back to Denmark.

With the death of his father, King Frederick, Christopher assumed the throne. Although we will never know with certainty what caused Christian's hatred of Tycho, we do know with certainty that Christian did in fact hate Tycho. Perhaps it was the rumors of a romantic liaison between Tycho and the Queen, Christian's mother, that was the source of Christian's enmity. There are rumors to this day that Christian even saw Tycho and his mother in the act.

It's all unverifiable rumor, but it certainly would explain Christian's subsequent bludgeoning of Tycho. Christian was 11 years old when Frederick died. A regency council governed the kingdom until he came of age in 1596, when he was officially crowned king. And what did Christian do with his authority? In 1597 Christian stripped away Tycho's rights to Hven and with it his rights to the income collected from the peasants on Hven. As though this was not enough, Christain forced Tycho into exile and burned his observatory to the ground. In 1597, shortly after Kepler published *Mysterium Cosmographicum*, Tycho left the grave-sight of his pet moose.

In 1598, after visiting several potential sponsors, Tycho set his sight on Prague. Tycho had hoped that the supreme ruler of the Hapsburg empire, Emperor Rudolph, would enlist Tycho's services as the court astronomer. This was not a long-shot. Tycho had the reputation as Europe's preeminent astronomer and Rudolph – a superstitious figure who employed a coterie of astrologers, soothsayers, and even Jewish kabbalists [1]– was keen on hiring Tycho. Tycho and Emperor Rudolph concluded their negotiations that year. Tycho did not take up residence until the following year, 1598. In the interim he had to arrange for the transport of the equipment and personnel that he managed to remove from Hven.

While returning to Prague in 1599, Tycho obtained possession of a recently published book that was sparking some excitement among the community of astronomers. The book, *Mysterium Cosmographicum*, left a positive impression. Whether or not he bought into Kepler's vision of God's design is unknown and irrelevant. Tycho most certainly recognized a formidable mathematician with talent beyond any mathematician that he ever encountered.

An encounter between Kepler and Tycho would occur in February of 1600, seven months after Tycho settled into his position as Imperial Mathematician. Due to a deportation order of non-Catholics, Kepler found himself unemployed and homeless. The order applied to the Lutheran community of Graz where Kepler was a school teacher. Bless their hearts the authorities behind this edict did provide the opportunity for the non-Catholics to avoid deportation by giving up their identity and converting. Unlike many of Kepler's colleagues who went along and converted, Kepler upheld his beliefs and confronted unemployment and poverty rather than going along. One wonders how his wife and children responded to Kepler's steadfast upholding of his convictions.

With the hope of obtaining employment, Kepler wrote a letter to Tycho requesting an audience. Tycho extended an invitation; most likely he was eager to meet the author of the book that had impressed him. While the stars eventually aligned in harmony, the beginnings indicated a violent collision.

Tycho invited Kepler to work for him, granting Kepler room and board at his castle, but no cash. Kepler was an unpaid servant. The conditions ate at Kepler who could not contain his anger. The commoner confronted the prestigious nobleman and a shouting match ensued, followed by Kepler's immediate departure.

Had Kepler been a merely capable man, the departure would have marked a permanent divorce. But Tycho

[1]Oi veys mir, Jewish kabbalists in the Catholic court of the Holy Roman Empire!!!

knew that although Kepler was a commoner, he was most uncommon. At Hven, Tycho collected the most comprehensive set of heavenly observations available to anyone in Europe[2]. He needed Tycho's skills to uncover the configuration of the universe that the observations held secret.

After a cooling off period, Tycho offered acceptable employment terms and Kepler returned to Tycho's castle. A little further along, their relationship improved. They bonded and aligned, sort of. But destiny did not permit it to last long at all. In 1601, on his death bed Tycho bequeathed his most precious possession, his observations, to Kepler. Tycho requested, "Do not let me to have lived in vain". Tycho left his observations in the right hands. Kepler acceded to Tycho's wishes beyond what Tycho could have imagined. Indeed, had the heavens not conspired to introduce Kepler to Tycho, history might have forgotten Tycho.

Although Tycho verbally bequeathed his observations to Kepler, Kepler was not the legal inheritor. What Kepler did inherit was Tycho's unbelievable story, and Kepler lived that story in Tychonian fashion. The story begins with Kepler's ascension to the Imperial Mathematician soon after Tycho's death.

The legal inheritors of Tycho's estate, including his observations were Tycho's daughter and by extension her husband. There was a foul relationship between the husband and Kepler. Kepler knew that it would not be long before the husband pressed his rights to take possession of the data. Fortunately, as the Imperial Mathematician Tycho had access to the data, but also knew he had to act quickly. With his eyes on Mars, he either absconded with the Mars observations.

## 5.2 The Battle for Mars

Kepler rightfully believed that the Mars observations were the key to understanding the motion of the planets. He brashly thought that he would crack the nut in short order, months as opposed to years. As with many wars that will with certainty end in swift victory, this one lasted for years; eight years to be precise. Aside from the intellectual battle with Mars, Kepler confronted the legal battle that deprived him of access to the Mars data for two of the eight years.

The story of Kepler's triumphs begins with his failures. An obvious point, but one worth stating is that Kepler was a Copernican, fully heliocentric. His plan for Mars was to expose its journey around the Sun; no failure here. However, Kepler the scholar was aware of the Greek tradition that the circle was central to the movement of the heavens. Kepler did not share the pagans' theology, but he accepted that the circle, which no other geometric form rivals in symmetry and thus perfection, must be central to God's design.

Kepler, the mathematician fully comprehended the Ptolemaic universe, where circles rule. He knew of all of Ptolemy's tricks that nudge the circular orbits of Ptolemy's geocentric universe in the direction of the observations, the equants, the epicycles, the offset deferents. In Kepler's mind, Ptolemy was right. The circle was God's design and those features that Ptolemy imposed would lead to the correct orbits. But also, Ptolemy was wrong; those orbits were not about the Earth but circled the Sun. Kepler would correct the Ptolemaic mistake using the correct Ptolemaic methodology in short order.

What a disaster. Two years into the project, well beyond his proclaimed delivery date, he was nowhere closer to his objective. Then the inevitable happened. Tycho's legal heirs discovered that Kepler was using the Mars observations that were legally theirs and were determined to put an end to it. A lawsuit ensued and the judge ruled in the heirs' favor depriving Kepler of the data. Was this a disaster or a gift?

For two years, Kepler turned his energy toward other endeavors. The first, which would have certainly won Kepler a Nobel prize had there been a Nobel prize at the time, was an inquiry into optics. Kepler's works gave

[2]The Chinese likely had an even richer set of observations that they created throughout centuries of dynastic support.

a scientific explanation of the telescope. Additionally, once recovering the Mars data, Kepler would be able to apply his studies toward correcting optical distortions in Tycho's Mars observations.

We don't know why or how, but Kepler got his hands on Apollonius' treatise on the ellipse. The treatise includes an overview of Archimedes' works. The two most brilliant of all Greek scholars uncovered more properties about the ellipse than one might imagine could exist. In fact, in the intervening years, around 1800 years, nobody discovered any additional properties, until Kepler.

The property that Kepler found is the one that every student in the world learns in their geometry class. One can form an ellipse by placing a loop loosely around two fixed points and extending the loop in all possible directions. The fixed points are the foci of the ellipse and the extension in all directions forms the ellipse. Kepler knew everything that Archimedes and Apollonius knew and more. He was the indisputable world's leading authority on the ellipse. Because of the unwelcome diversion from Mars, Kepler became the expert on the subject he would rely upon to discover God's design.

## Confounded

In 2004, Kepler negotiated a successful peace settlement with Tycho's daughter and her husband that permitted him access to Tycho's Mars observations. He had new plans of attack. As noted above, one was to use his new understanding of optics to correct observations distorted by optical effects of the atmosphere. He also used the corrections along with clever geometric analysis to determine the relative positions of both Earth and Mars with respect to the heliocentric Sun at their corresponding observation points.

The main thrust of his renewed attack was to remove a constraint that simplified Ptolemy's work. Ptolemy constrained the deferent's center to lie at half the distance between the Earth and the equant[3]. With this constraint, Ptolemy developed the equations that allowed him to determine the equant and the deferent. The equations are not solvable, one can only approximate their solutions. Ptolemy developed an ingenious method to approximate the solutions that is now a standard feature in applied mathematics, the iterative method[4]. Ptolemy required four iterations. Removing the constraint complicates the equations, compounds the computational effort at each iterate, and necessitates an increase in the number of iterations before reasonable accuracy is assured. Kepler required 70 iterations and in his book *New Astronomy*, Kepler filed a complaint about the required effort.

Eight minutes...Eight minutes was enough for Kepler to abandon years of work. Kepler had determined all the equant, and deferent parameters that would be best fits to the observations. He was at the climactic moment of triumph. Just one final validation, a cross-check against the observations and victory was his. But there it was, an eight minute discrepancy, $2/15^{ths}$ of a degree with one of Tycho's observations. Eight minutes that changed the world.

I (the human author) have imagined what I would have done if I were in Kepler's position. Suppose I devoted a lifetime to astronomy and studied everything from Ptolemy to Copernicus. I labored for five years to determine the orbit of Mars holding down two full time jobs; duties as the Imperial Mathematician[5] and my devotion to

[3] The deferent is the large circular path around which a planet's epicycle travels. In Kepler's model, the deferent becomes the path of the planet itself (no epicycles). The equant is an off-center point from which Ptolemy assumed uniform angular motion. Since this point does not coincide with the deferent's center, the planet's speed along the deferent varies, making the motion appear more complex.

[4] The method requires one to at first place an approximate solution into the equation and use the equation's output to improve upon the approximation. Repeat this process iteratively until the difference between the equations's outcome and the approximation's outcome is immaterial.

[5] Rudolph was demanding. His superstitious nature and fervent belief in astrology required a reading of the stars for even minor decisions. Kepler complied.

the discovery of God's design. I applied all known techniques and after an exhaustive effort had a solution.

Okay, I find the eight minute difference between my solution and one of Tycho's observation, but eight minutes...come on. Hold your thumb out at arms length. The breadth of the circle that the thumb cover is about one degree. Divide that thumb into 15 parts and take two of them. That's about knife's edge. One observation, a knife's edge off would most likely be no obstacle for me.

There would be plenty of ways to explain the difference. Tycho entered thousands of observations into his record. He not only compiled observations for Mars, but for all of the planets and 777 stars as well. It is beyond human capacity to enter every observation accurately. There are many sources of error. It could have been a recording error. It could have been an error in the reading of the sextant. It could have been a siting error, a misalignment between the observer, the sextant, and Mars. What about the weather? Perhaps on the night of that particular observation, the weather did not afford the observer the requisite time to complete an accurate observation. The temptation to accept these excuses would have likely overcome me. I would have declared victory and sought out the oncoming praise.

What's more who could have challenged me? Nobody on the planet had anything better to offer. This was mankind's best description of Mars' orbit. But Kepler's intellectual integrity was as rare as his intellectual gifts. The only man on Earth competent to challenge Kepler was Kepler himself. And he did.

We know in detail all of Kepler's efforts. How so? Throughout his work, Kepler recorded an explanation of his efforts, his interpretation of results, his next course of action. *New Astronomy* is part scientific compendium and part diary that includes personal details such as complaints about the monotony of some of the computations he undertook, in particular the aforementioned 70 iterations.

*New Astronomy* is a distinctive look into a scientist's discovery. All other scientific writings that I (biological author) have read present their findings in finalized form. There is a direct path from problem statement to conclusion that is logical and concise. Had Kepler followed the standardized format as developed by the Greeks, *New Astronomy* would be void of the eight minute discrepancy, it would be void of any of Kepler's Ptolemaic efforts, it would be void of countless other failures that Kepler openly reveals. It would be void of the process by which science truly progresses, repeated failures followed by success. The following translations attests to Kepler's honesty with himself and openness to public review.

At one point Kepler expresses his confidence in his approach.

*You see then, Oh studious reader, that the hypothesis founded by the method developed above, is able in its calculations to account for not only the four observations upon which it is founded, but also able to comprehend the other observations within two minutes...I therefore proclaim that the achronycal positions displayed by this calculation are as certain as the observation made with Tychonic extants can be.*

One further validation was necessary. Kepler needed to adjust his model to account for a small angle between Earth's plane of revolution about the Sun (the ecliptic) and that of Mars. His writings indicate that he felt that the small angle would have immaterial impact on the results. Victory was not only at hand, its smell was within his nostrils. But then he writes...

*Who would have thought it possible? This hypothesis so closely in agreement with achronycal observations is nonetheless false.*

And later on he explains.

*Since the benign benevolence has vouchsafed us Tycho Brahe, a most diligent observer, from whose observations the eight minute error in this Ptolemaic observation is shown, it is fitting that we with thankful mind both acknowledge and honor this benefit of God. For it is in this that we shall carry on to find at length the true form* *of the celestial motions, supported as we are by these arguments to be fallacious. In what follows, I shall myself, to the best of my ability, lead the way for others on this road. For if I had thought that I could ignore eight minutes in longitude, in bisecting the eccentricity, I would already have made a correction in the hypothesis found in chapter 16. Now because they could not have been ignored, these eight minutes alone will have lead the way to the reformation of all astronomy, and have constituted the material for a great part of the present work.*

Kepler needed something else. We romanticize the past and aggrandize those who came before us. Kepler did as well. Apollonius and Ptolemy were two of Kepler's larger than life heroes and so once more he turned to them. For the purpose of explaining the apparent retrograde motion of Mars against the stars, Apollonius proposed the epicycle. Ptolemy accepted the epicycle and configured its dimensions and speed so that planetary orbits matched observations. Because heliocentric orbits do not require epicycles to explain retrograde, Kepler initially rejected them as superfluous. Still, the epicycles were the last offerings of his heroes, so they were his next stop.

It was a nearly impossible year-long slog before Kepler introduced another device. This one was his own discovery independent of the ancients. He was the first to calculate the relative distances between Earth, Mars, and the Sun. Kepler loved to find relations within data. The difference between genius and really, really smart is that an outsider can recreate the thought process of a really, really smart individual. But maybe not even the genius in question can explain how they arrive at their insights[6]. A really, really smart individual could have calculated the distances. But only a genius could have spotted the insight that had eluded the ancients.

Kepler proposed and proved what history would designate his Second Law. The translated words are"That the radius vectors describe areas proportional to the times." We paraphrase Kepler in a manner that's somewhat easier to interpret; a planet sweeps out equal areas in equal times. Fig 5.1 illustrates the concept. The planet proceeds around the orbit in a counterclockwise direction. Positions where the planet enters and exits the respective areas are designated by dots along the orbit. The time difference between the respective entry and exits is equal. Kepler's Second Law states that the respective areas are also equal.

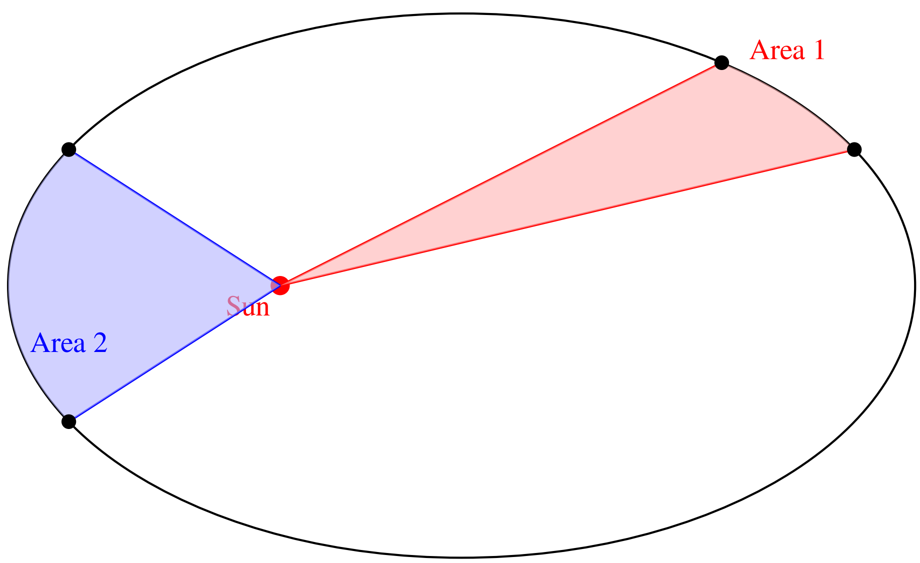


Figure 5.1: Illustration of Kepler's : A line connecting a planet to the Sun sweeps out equal areas in equal times.

This is all rather confusing. The Second Law came first and the First Law came second. Kepler's goal was

[6]While attempting to embarrass Ramanujan, a Professor at Oxford called him to the board to solve a problem requiring mathematics beyond the student's knowledge. Ramanajun instantly discovered the technique and solved the problem. When the professor asked, "Where did you learn that?" Ramanujan responded, "It just came to me".

to use the Second Law as a means to discover the shape of the orbit, which is the First Law. The confusion reflects Kepler's confused state of mind.

### The Slap

Have you ever stared at the obvious only to draw the wrong conclusion. Then when the moment of revelation finally arrives you slap yourself and maybe even call yourself an idiot out loud. If this experience sounds familiar, you are in good company. It happens to geniuses, even Kepler and pretty much everyone [7].

With his new tool, Kepler committed himself to configuring the parameters of the Ptolemaic orbit; fix them so that the orbit obeys the data. The problem was that the required calculations were exceedingly difficult. He needed some device that would ease the computational burden. Once again, he reached out to two of his heroes, Archimedes. [8] and Apollonius. As mentioned earlier, Kepler read Apollonius' treatise on the ellipse and became the world's leading authority. The ellipse would be a good proxy for the orbit; not the true orbit which was Ptolemaic, but a device that would assist with required computations.

Kepler configured an initial ellipse that was his best fit to an orbit that had been his best effort since his epicyclic quest. He then positioned Mars across the ellipse in a manner that conformed with his Second Law. This provided not only the shape of the orbit, but the travel times between points on the orbit. These calculations were difficult and time consuming; they took months, several months, not days or weeks.

Kepler compared the results against uniform motion about a circle and noted that several critical observations of Tycho's fell between the corresponding observations of the circular and elliptic orbits in sequence. Kepler drew the conclusion that had been in his face for months and took his slap. In *New Asrtronomy* he records the reasoning that lead him to the ellipse.

*The circle of chapter 43 errs in excess, while the ellipse of chapter 45 errs in defect. And the excess of the former and the defect of the latter are equal. But the only figure occupying the middle between the circle and an ellipse*

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[7] Einstein took the slap when he set his famous Cosmological Constant to a value that fixes the size of the universe. He famously made a public confession with the regret that he should have predicted an expanding universe. This was not Einstein's only slap. It's a part of the human experience that even geniuses cannot escape.

[8]

As a historical mathematical figure of ancient Greece, Archimedes is unique. We know next to nothing of his contemporaries. There are few personal stories of Eudoxus, Aristarchus, Apollonius, Hipparchus, or Ptolemy. And yet personal anecdotes, whether true or not, about Archimedes abound.

We have heard that he was so excited by his discovery of the Law of buoyancy that he jumped out of his tub and onto the street naked yelling "I found it" (Eureka). Also, upon elucidating the Law of Levers he claimed he could lift the earth if his lever were sufficiently long. Then there's the story about his use of an array of mirrors to incinerate the Navy of oncoming Roman invaders, certainly pure fiction. More believable is his invention of catapults that allowed the Syracusans to accurately bombard the approaching Romans; Archimedes developed catapults calibrated to the distance of the target. Somewhat believable is the story of his application of the Law of Levers to hoist Roman naval boats out of the sea.

It is said that the Syracusans perched the lever upon a cliff overlooking the sea. On one end of the lever was a rope attached to a special claw that Archimedes designed. The other end was attached to a rope and a weight. The claw would grasp a boat at a landing site. After the claw secured itself to the boat, the Syracusans released the opposing weight resulting in the boat's dangling helplessly in the air.

The circumstances of his death along with his dying words have also been kept for posterity. The words were, "don't step on my circles". And the circumstances? Roman soldiers who after a three year siege finally breached the city walls interrupted Archimedes while he was focused on a geometric problem. He had drawn circles into a sandbox in his home and apparently the soldiers had less respect for geometry than Archimedes. They answered Archimedes' scolding by thrusting a spear into him.

The stories don't end with Archimedes' death. It is said that out of respect, Romans gave Archimedes an honorable burial. The tombstone's epitaph was an engraving of a sphere and a cylinder. By mapping the sphere onto the cylinder, Archimedes was the first to discover the formula for the area of a sphere; a result he was most proud of.

Twenty two centuries later, Einstein became both a scientific and social phenomena. Perhaps Archimedes and Einstein are the only two individuals in history to achieve such legendary status.

*is another ellipse. Therefore, the ellipse is the path of the planet.*

The discovery of a letter to a friend provides the explanation for Kepler's hesitance to consider an ellipse as the actual orbit. In that letter Kepler shows reverence for his heroes. He communicates that surely Apollonius or Archimedes would have discovered the ellipse as the orbit if that were the case. Thus he clung to the devices that Ptolemy had employed.

### Victory, No Questions Asked

Kepler had finally severed himself from his Ptolemaic umbilical cord. He knew exactly what he had to do; perform a two step process that would first determine the configuration of the ellipse and then determine the transition time between points on the ellipse using the Second Law. *New Astronomy* was his first effort. It appears that he later recognized that the legal path through the Second Law) could both configure the geometry as well as the journey. But in *New Astronomy*, Kepler separates the two.

A circle is a special ellipse. To specify a circle, one needs only to designate a single value, its radius. Once the radius is set all the other points fall into place. For a general ellipse, setting the position of the focal points and the length of the major axis is sufficient for determining all other points. For those who fondly recall the Keplerian drawing of an ellipse using the loop and peg method, the pegs are the focal points and the length of the loop gives the length of the major axis.

To configure the ellipse, Kepler turned to his failure, that eight minute failure. Within the ecliptic, the failed model was a spectacular success. And so, he used it as a false model to set the parameters of the true model. Kepler set the major axis equal to the diameter of the circular orbit of his model. Then Kepler set the Sun at the point opposite the equant on the major axis. In this manner the equant and the Sun were symmetrically positioned about the center of the major axis.

Now he was ready to apply the Second Law and specify transition times. Ideally, this is how it would go. Start at the aphelion, the furthest point from the Sun. For any other point on the elliptic orbit, calculate the area on the sector between the aphelion, Sun, and point of interest. This informs you of the transit time between the aphelion and the point of interest.

Sounds simple enough, but more than four centuries after the publication of Kepler's *New Astronomy*, we are unable to carry out this ideal computation. There is no general formula for the area of a sector of an ellipse. Using a computer, it is possible to make extremely accurate numerical approximations. But Kepler had to make these approximations by hand; they are extremely tedious and time consuming.

He chose three points and for those three points went through the necessary numerical computations. Everything checked out for the three points. What next? Was there the scrutiny that he demanded for the failed model? Was there the validation and honest confrontation with other points in the expansive Mars dataset? Was there a meeting between Tycho's data, that God himself vouchsafed, and Kepler's ellipse? Two decades after the publication of *New Astronomy* Kepler published the *Rudolphine Tables*. That publication opens Kepler's works to scrutiny and Kepler's victory is obvious. However, within *New Astronomy*, Kepler simply declares victory without putting the ellipse to the test.

Every saint has their flaw. Kepler, once discovering beauty obsessed over his discovery. In this case, Kepler convinced himself that he had found God's design. The failed model deserved Kepler's scrutiny. It was no more than a parameter selection process. Of course after matching parameters, it is necessary how well the model actually comports with the data.

Kepler believed his ellipse went beyond a data matching exercise. In *New Astronomy* Kepler gives a physical explanation of the forces radiating from the Sun that cause a planet to orbit about the Sun in an ellipse. It was in the physical cause that Kepler found God's design. The ellipse was merely a product of the physical cause. Configuring the ellipse only requires three observations. Once three points on the ellipse match the observations, all the points must do so.

We applaud Kepler for his discovery of the ellipse as the planetary pathway. Ask your neighbors what Kepler achieved and there is a good chance they will answer, he found the ellipse. Ask your neighbor about Kepler's explanation of the physical forces causing the ellipse; you'll most likely get blank stares. His ellipse survives and we know of his contribution, because he was dead right. His explanation of the forces acting on a planet never took hold, because he was dead wrong.

Kepler would live to see the success of the *Rudolphine Tables*. He joined God knowing that God approved. However, concerning his other idea, perhaps God gave him a bit of a ribbing.

## 5.3 The Struggle

The orbits of the planets had been a source of scientific inquiry for nearly two millennia. During that time all efforts to provide a convincing description for the motion of the heavens failed. Eudoxus failed, Ptolemy failed, efforts to tweak the Ptolemaic universe failed, Copernicus failed. Their models were brilliant, but they didn't pass the scrutiny of observations. The data did not lie, so the heavens remained unexplained—until Kepler.

Kepler's conquest restored order to the heavens. Planets float through space about their elliptic orbits without disturbance, indefinitely through time.

Kepler's ellipse is a solution to what is known as the two body problem. Allow two bodies such as the Sun and a planet to interact with one another. If one body is much larger than the other; i.e. the Sun, then it is nearly stationary and the trajectory of the smaller body about the larger one is an ellipse.

There is a natural extension of the two body problem known as the three body problem. Unlike the two body problem there is no tidy solution to the three body problem. A smaller body's motion can get tangled in an endless pretzel that encompasses two larger bodies. The larger bodies jerk the smaller body about causing random and abrupt shifts in the smaller body's trajectory. At one moment it appears to orbit about one of the larger bodies only to be seized by the other larger body without notice. The abrupt shifts continue indefinitely through time.

This is not a Keplerian ellipse orbiting predictably and peacefully without disturbance. This is chaos. Kepler's journey through life would not be along the peaceful, predictable path of the ellipse that he discovered. Kepler's journey was akin to that of the smaller planet in the three body problem. Forces larger than himself could prevail upon him and upset his course at any moment.

We have already mentioned the unfortunate circumstances of Kepler's childhood. In his adult life, death was a frequent presence. His first wife died in 1611, two years after the publication of *New Astronomy*. Among his children, survival to adulthood was a little over a 50-50 proposition; five succumbed, six survived to adulthood.

Kepler lived at a time where a toxic mix of politics and religion poisoned much of the European continent. Kepler's position as a Lutheran in Catholic court made him a target for both sides to shoot at. In 1911, Rudolph forfeited his imperial crown to the more strident younger brother, Mathias. Wanting no Lutheran presence in his court, Mathias forced Kepler's departure. In 1613, Kepler resettled in Linz, where he remained in Mathias' services, but received little of the promised payment for those services.

Punishment from the Lutheran community was more severe. Recall, while in Graz, Kepler remained loyal to Lutheran theology and refused to convert to Catholicism. Rather than convert to Catholicism, Kepler continued with the Lutheran Eucharist while having little bread to feed his family. In return for his loyalty, in 1612, the

Lutheran leadership excommunicated Kepler. Excommunication was the result of a targeted mission by those who sought to discredit Kepler. Why? Why not!

And as if excommunication was not enough of an insult, in 1615, the Lutheran community in Kepler's home district of Wurtemberg brought Kepler's mother to trial charging her with witchcraft. Throughout the six year trial, Kepler assumed the role of his mother's legal council. While Kepler's keen mind saved his mother from the pyre, the incident could not have been easily dismissed and most likely left a burning scar on Kepler's psyche. As for Kepler's mother, she died in April 1622 shortly after her release from prison in October 1621.

Religious and political disputes inevitably resulted in war. The Thirty Years War began in 1618 while Kepler was in Linz. The outcome of the war was a political realignment of Europe in which Spain lost its position as the dominant power. Spanish loss of authority garners no sympathy, no tears, but the loss of some six to eight million people, mostly civilian casualties, touches the heart. Kepler was right in the heart of it.

Linz was under the dominion of the Catholic Habsburg's. The population was predominantly Catholic with Protestant enclaves on the outskirts. Kepler resided among the Protestants. In the early 1620's and 1626, Protestant forces attempted to lay siege to Linz and bombarded its Catholic districts. Kepler was a first hand witness to the war and described its impact on his neighbors and his own family. Life was at stake as civilians were caught in the battle's cross-fire. The seige resulted in food shortages. Another outcome was further persecution of the Lutheran community which alongside their empty bellies left an emptiness in their spirits. The 1626 battle for Linz left Kepler in an untenable situation. He moved his family once again, eventually settling in Sagan, Poland.

Despite promises from Mathias to pay Kepler for his publication of *The Rudolphine Tables*, Kepler did not receive his entitlement. In 1630, for the purpose of petitioning for unpaid salary, Kepler traveled to Regensburg. Weather conditions were not favorable and Kepler fell ill. The disease progressed and a Lutheran minister was called to Kepler's bedside. Following the protocol of the Lutheran Church, the minister denied the excommunicated Kepler of his last rites. Instead, the minister left a stinging inquiry. He asked Kepler how he expected to enter heaven.

Kepler had a clean conscience. He lived a moral and faithful life. With full confidence in his future he replied that his salvation was in the hands of Christ. Then he made his final move alongside God.

## 5.4 The Legacy

Despite the inhospitable circumstances that engulfed Kepler, he had an incredibly productive life. It is particularly impressive that Kepler was able to continue his lifetime contributions to science after his expulsion from Prague when death, instability, food scarcity, and war all swarmed about.

1. **Mysterium Cosmographicum** (1596)
   Kepler's first major work, in which he proposed that the spacing of the six known planets could be explained by nested Platonic solids. Although incorrect, the work marked his lifelong search for mathematical harmony in the cosmos.

2. **Paralipomena ad Vitellionem** (1604)
   A work on optics, this book introduced Kepler's law of refraction and explained how images are formed in the eye, laying the foundation for modern optics. It also included an explanation of how convex and concave lenses form images—important for the later development of telescopes.

3. **Astronomia Nova** (1609)
   This groundbreaking astronomical work introduced Kepler's first two laws of planetary motion:

a) Planets move in elliptical orbits with the Sun at one focus.

b) The line connecting a planet to the Sun sweeps out equal areas in equal times.

Based on meticulous data from Tycho Brahe, this work concluded the nearly two thousand year old debate concerning the motion of the planets.

4. **Dioptrice** (1611)
A continuation of Kepler's work in optics, this treatise explained the principles of how lenses form images and described the **Keplerian telescope**—a design that replaced the Galilean telescope and became standard in astronomical observation.

5. **Stereometria Doliorum Vinariorum** (1615)
In this treatise on the geometry of wine barrels, Kepler used early integral methods to calculate volumes. This was one of the first significant applications of infinitesimals and laid groundwork for integral calculus.

6. **Harmonices Mundi** (1619)
Here, Kepler published his third law of planetary motion:

$$\frac{T^2}{R^3} = \text{constant}$$

He also explored harmony in geometrical figures and planetary motion, integrating music theory, astronomy, and mathematics.

7. **Epitome Astronomiae Copernicanae** (1617–1621)
One of Kepler's most influential works, this multi-volume textbook systematically presented and defended the heliocentric model and included all three of Kepler's laws of planetary motion.

8. **Rudolphine Tables** (1627)
A set of astronomical tables compiled from Tycho Brahe's observational data and Kepler's laws. These tables improved predictive accuracy and remained in use for more than a century.

9. **Somnium** (written before 1630, published posthumously in 1634)
A visionary narrative often considered the first work of science fiction. It described a journey to the Moon and included realistic astronomical concepts embedded within a fictional framework.

These works directly shaped the trajectory of physics and mathematics, serving as essential precursors to some of the most transformative ideas in scientific history. Kepler's pioneering work in optics, particularly his understanding of refraction and his formulation of the inverse square law for light intensity, laid a conceptual foundation that influenced Pierre de Fermat in developing his principle of least time, a fundamental variational approach that redefined the behavior of light. Meanwhile, Kepler's three laws of planetary motion, derived from Tycho Brahe's observations and grounded in rigorous mathematical reasoning, provided the empirical and conceptual basis upon which Isaac Newton constructed his laws of motion and universal gravitation. Newton's proof that planetary orbits must be ellipses under an inverse-square force law was a direct confirmation of Kepler's astronomical insights. Furthermore, Kepler's lesser-known yet significant work on calculating the volume of wine barrels, detailed in his *Stereometria Doliorum Vinariorum*, tackled the problem of measuring irregular volumes. This work influenced the later development of mathematical analysis and anticipated key ideas that Newton and Leibniz later formalized into calculus. In these ways, Kepler's scientific legacy is not merely inspirational but foundational, forming a critical bridge from Renaissance natural philosophy to the precise, quantitative sciences of the Enlightenment.

## 5.5 Archimedes' Area, Petiscus' Precision, and Kepler's Calibration

As bait to gain Emperor Rudolph's support for Kepler's investigation into planetary motion, Kepler promised to author and publish the *Rudolphine Tables*, a collection of Tycho Brahe's observations along with recipes for forecasting the positions of the planets[9]. Kepler also provided forecasts of his own. Emperor Rudolph would have been pleased had he lived to see the tables.

Kepler published *New Astronomy* in 1609, but did not publish the tables until 1629. The years of chaos as well as lack of promised funds contributed to the delay. Rudolph's platoon of astrologers, sooth sayers, and kabbalists could only stand by and helplessly watch as he passed away in 1612. Rudolph may have well cursed Kepler on his death bed for not providing the tables. Perhaps they would have allowed for a reading of the heavens that could have prevented his demise.

Kepler relied upon the Second Law to configure the ellipse and the journey of several planets. For the purpose of determining the necessary sector areas, Kepler turned to a method discovered by Archimedes . Below,we go into further details of the Second Law and the area calculation. The objective is to determine the longitude of a planet at any time within a heliocentric system.

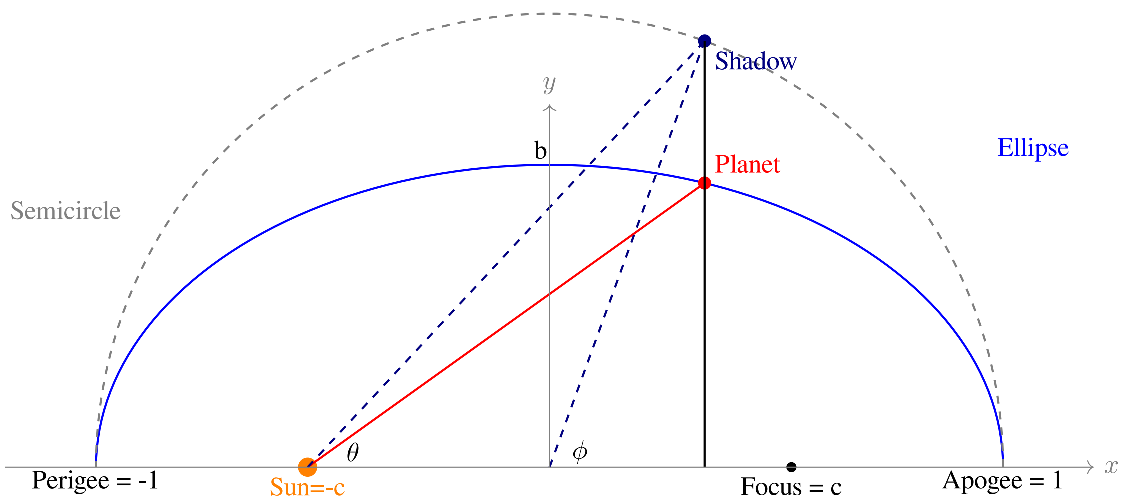


Figure 5.2: Illustration of Kepler's adaptation of Archimedes' method: approximating the area swept by a planet using a shadow construction on a reference semicircle.

Figure 5.2 gives a single position of a planet along its elliptic path. The angle $\theta$ is the heliocentric longitude of the planet. The objective is to determine $\theta$ as a function of time; the planet travels counterclockwise about the ellipse. Note that the longitude is independent of the units used to measure distance. For ease of computation, we set one unit of measurement equal to the length of the semimajor axis. The length of the semiminor axis is $b < 1$.

Accompanying the planet is a shadow that lies upon a circle of radius one. The shadow's position always lies directly above the planet's position, the $x$ coordinates of both the planet and the shadow are the same. A property of the ellipse is that the $y$ coordinates are related by the equality, $y_{planet} = by_{shadow}$. With this geometry, Archemedes demonstrates that the area of the planet's sector and that of the shadow's sector are related by the equation[10]

[9]This has similarities with Guo Shoujing's Shoushi Li. There is a recipe for determining future events

[10]The equality of areas is due to the property that for any value of $x \in (-c, 1)$ all line segments between the upper and lower

$$A_{planet} = bA_{shadow}$$

Figure 5.3 illustrates the relevant sectors.

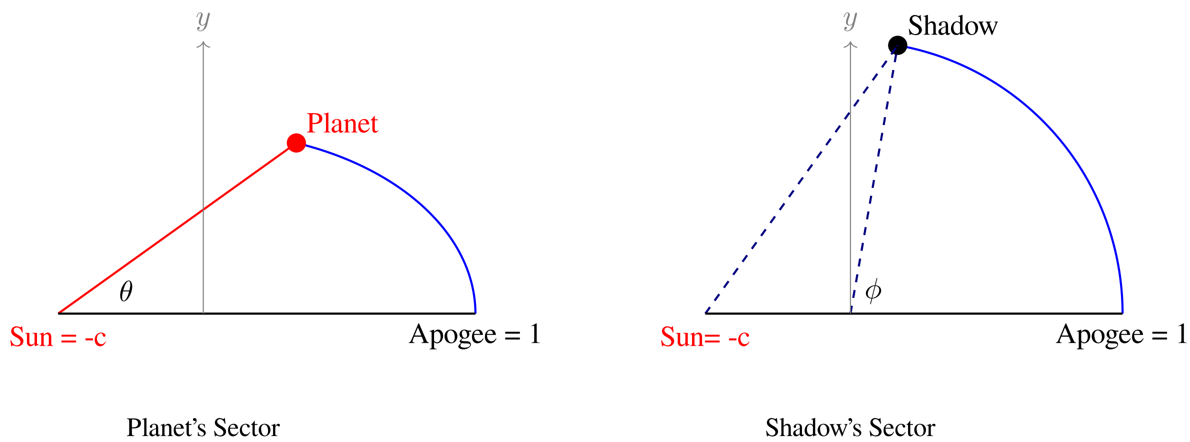


Figure 5.3: Swept Areas: Planet's Sector and Shadow's Sector

As if anticipating Kepler's needs nearly two millenia before Kepler, Archimedes' equality is tailored for the Second Law. The Second Law states that the time required to travel from the apogee to the planet's position is proportional to the area of the planet's sector. Archimedes demonstrates that the the area of the planet's sector is proportional to the area of the shadow's sector. Therefore, the time required to travel from the apogee to the planet's position is proportional to the area of the shadow's sector. Kepler can reduce the problem from finding the area of the planet's sector to that of finding the area of the shadow's sector. In honor of Kepler's efforts, we outline the tedium of the necessary calculations.

Note that the shadow's sector is formed by two regions, a triangle and a pie slice. The triangle has vertices Sun, shadow, and the origin where $(x, y) = (0, 0)$. The pie slice has edges that run between the origin and the apogee, the origin and the shadow as well as an arc running from the apogee to the shadow. The following formula gives the area of the shadow's sector.

$$\begin{aligned} A_{\text{sh}} &= A_{\text{T}} + A_{\text{ps}} \\ &= \frac{1}{2}[c\sin(\phi) + \phi] \end{aligned}$$

where

- $A_{\text{sh}}$, $A_{\text{T}}$, and $A_{\text{ps}}$ are the areas of the shadow sector, triangle, and pie slice.
- $\phi$ is the angle measured in radians ($2\pi$ radians $= 360°$).

Applying the Second Law gives the following relation between the area of the sector and the time required for Mars to travel from the apogee to a point on the ellipse.

$$t = \frac{k}{2}[c\sin(\phi) + \phi]$$

boundaries of the sectors are in proportion to one another. The constant of proportionality is $b$. This is known as Cavalieri's Theorem, circa 1635, but Archimedes preempted Cavalieri by nearly two millennia.

where

- $A_{\text{sh}}$, $A_{\text{T}}$, and $A_{\text{ps}}$ are the areas of the shadow sector, triangle, and pie slice.
- $\phi$ is the angle measured in radians ($2\pi$ radians $= 360°$).

In the above equation, where where

- $t$ represents the time required for the shadow to travel from the apogee to the angle $\phi$.
- Archimedes shows that $t$ is also the time required for Mars to travel from the apogee to the longitude $\theta$.
- $k$ is the constant of proportionality.

Let the unit of time that we work with be one Martian year, which in the *Rudolphine Tables* Kepler sets to 686.95 days. (For comparison, the currently accepted values is 686.98 days; the accuracy of Kepler's estimate is a testament to both Kepler's analytic skills and Tycho's observational skills.) Accepting 685.95 days as the unit of time, the angle $\phi$ sweeps through $365° = 2\pi$ radians in one unit of time. With these values for $t$ and $\phi$ we can solve for the constant of proportionality, $k$.

$$1 = \frac{k}{2}\left[c\sin(2\pi) + 2\pi\right] \quad = k\pi$$
$$k = \frac{1}{\pi}$$

In order to determine the longitude of Mars at any time, Kepler needs to configure the ellipse which is set by the orientation of the major axis and the focal distance. With the correct orientaion and focal length he can position the Sun within the ellipse.

The next section gives a description of Tycho's data. Here we note that the data does not provide a longitude with respect to the apogee, but one is able to ascertain changes in longitude from the data. Using Tycho's observations of time differences at opposition as well as the differences in longitude it is possible to use the above identities to orient the axis and determine the value $c$. We leave this problem as an exercise for the motivated reader. The reader familiar with the properties of the ellipse is aware that the value $c$ fixes the semiminor axis, $b$.

We go on to present Kepler's solution to another problem. Given the time of passage from the apogee $t$, determine the longitude of Mars, $\theta$. This requires a two stage process, determine the angle $\phi$ and then determine the longitude $\theta$.

An equation for $\phi$ is available from the above identities.

$$t = \frac{1}{2\pi}\left[c\sin(\phi) + \phi\right]$$
$$\phi = 2\pi t - c\sin(\phi)$$

Consider the second equation as a solution for $\phi$. A chorus of objections arises, "this is a worthless equation, you have to know the value of $\phi$ before you can solve for it". The objection is correct, but Kepler is a magician. He can use the equation to find an estimate to any degree of accuracy.

Suppose that a value of $\phi$ is somewhat close to the correct solution. What happens if you place this estimate into the right hand side of the second equation and solve for a new $\phi$ on the right hand side? As it turns out, the new value is closer to the actual solution than the original value. What next? Update the new value by putting it into the right hand side of the second equation and solving for the right hand side. The updated value is closer yet to the actual value. Repeat the process and you can come as close as you want. When do you stop? When the difference between two iterates is immaterial.

The table below gives the iterates with the focal value $c = .09266$ and $t = 1/6$. In terms of time, $1/6^{th}$ of the martian year has passed. An assumption of constant speed for the shadow angle gets one pretty close to the actual position of the shadow. This puts the initial guess for $\phi$ at $\phi \approx \pi/3$, which is the first entry on the table.

| **Iterate** | $\phi$ in radians | $\phi$ in degrees |
|---|---|---|
| 0 | 1.04719755 | 60.0 |
| 1 | 0.96695591 | 55.40249241 |
| 2 | 0.97092750 | 55.63004821 |
| 3 | 0.97071916 | 55.61811093 |
| 4 | 0.97073006 | 55.61873542 |
| 5 | 0.97072949 | 55.61870275 |
| 6 | 0.97072952 | 55.61870446 |
| 7 | 0.97072952 | 55.61870446 |
| 8 | 0.97072952 | 55.61870446 |
| 9 | 0.97072952 | 55.61870446 |

Table 5.1: Table of $\phi$ values by iteration

After six iterations, the value does not change. This is as close as the computer's numerical precision allows. Four iterations gives precision rounded to the fifth decimal place.

I (the biological author), have a computer available and am able to kick out the table without doing a single computation on my own. Kepler had to perform these computations without the aid of a computer. Notably he had to determine the value of $\sin(\phi)$ at each iteration. Kepler owed a debt to Bartholomaeus Pitiscus who in 1612 published a table of trigonometric functions with the sine table at intervals of one arcsecond. That's 3600 entries per degree and Bartholomaeus Pitiscus' sine table goes from zero to 90 degrees. Poor Bartholomaeus died in 1613, shortly after the publication of his exhaustive work. Even with the assistance of Pitiscus, Kepler's calculations were equally exhaustive and exhausting.

Kepler's next step is to find the longitude. Using Figure 5.2, one arrives at the following relation between $\theta$ and $\phi$.

$$\tan(\theta) = \frac{b\sin(\phi)}{c + \cos(\phi)}$$

Additional consultation with Pitiscus along with a few more calculations allows one to determine $\theta$ once $\phi$ is known. Alternatively, just write a single line of code on the computer and read the result, $\theta = 0.89607$ radians $=$ $51.34127°$.

## 5.6 Kepler, the Data Scientist

Was Kepler a data scientist? Absolutely. This section shows that Kepler's program comports with the program of a modern day data scientist outlined in Chapter 2.

### Define the problem.

Determine the orbital pathways the planets in the solar system.

### Propose an input-output parametric model of the system.

Kepler's First and Second Laws are input, output systems. The inputs are Tycho's observations. A careful analysis of the data reveals the directions of the apogee and perigee which orients the major axis as well as determines the position of the Sun along the axis. Inputs are then the observations and an output is the configuration of the ellipse as set by the orientation of the major axis and position of the Sun (the focal distance).

The Second Law gives the journey, transit times along the planet's orbit. Using the procedure described in the previous section configuration of the ellipse allows for the determination of sector areas, which the Second Law equates with the output, transit times.

The process depends upon accurate orientation of the major axis. The observations provide directions and time of observation points. From this, Kepler has triangulation methods that allow him to determine the distances which combined with the directions, gives the location of a planet at each observation. Theoretically, one could determine the direction and distance of the apogee and perigee from this data and fit an ellipse to the data.

It is our belief that Tycho's dataset would not allow Kepler to carry out the program of the preceding paragraph with the necessary precision required to specify the ellipse. Additional data that Kepler needs, i.e. the angular sector across the sun's diameter, would have been very difficult to measure with precision. Using triangulation, Kepler did map out the locations of Mars. We suspect that this mapping provided a first estimate of the orientation of the major axis and the position of the Sun along the major axis. However, Kepler further refined the estimate using Tycho's observations of direction and timing along with the Second Law.

### Collect and organize data as inputs and outputs.

For Kepler, Tycho Brahe's observations are both inputs and outputs. They are used as inputs to calibrate his model and outputs to test results.

Tycho's publication *Historia Coelestis* provides a catalog of observations. The language is Latin and Tycho uses symbols to represent planets, constellations and other objects in the heavens. With some effort, the partners in this book have come to an agreement on the meaning of some entries. Below is one such entry.

| Latin Entry | English Translation |
|---|---|
| DIE 23. NOVEMB. A.M. | November 23, morning |
| Horologium verificatum est ad stellas. | The clock was verified using the stars. |
| Alt. Meridiana ♂ per Q. Tych. 41 25½ novo pin. | Meridian altitude of Mars by Tycho's quadrant: 41° 25½′ (new pinnule) |
| 41 25 vet. pinn. | 41° 25′ (old pinnule) |
| H. 7 M. 14½ | At 7 hours 14½ minutes |
| Spica orient. 18° 21′ | Spica rising (east): 18° 21′ |
| ♂ occident. 9 11 | Mars setting (west): 9° 11′ |
| Dift. aequat. 27 32½ | Equatorial difference: 27° 32½′ |
| H. 7 20½ | At 7 hours 20½ minutes |
| Cor ♌ occident. 32° 36′ | Heart of Leo setting (west): 32° 36′ |
| ♂ occident. 10 50½ | Mars setting (west): 10° 50½′ |
| Dift. aequat. 21 45½ | Equatorial difference: 21° 45½′ |
| Pone itaq; hic dift. aequator. 27 32½ | Record the equatorial difference: 27° 32½′ |
| Declinatio ♂ uno pin. 7 19⅔ Bor. | Declination of Mars: 7° 19⅔′ North (by one pinnule) |
| Altero pinnac. 7 19⅚ | With the other pinnule: 7° 19⅚′ |

There are several points of note.

- Fractional minutes are used for both time and arcminutes.
- Symbols designate planets and constellations. The symbol for Mars is (♂). The symbol for Leo is (♌).
- The Heart of Leo refers to the star Regulus which resides in the constellation Leo.
- Longitudes are given with an east-west orientation. For example Spica rising (east) $18° \ 195/6'$ indicates that the longitude of the star Spica[11] is at $18° \ 195/6'$ east of a marker that indicates the Vernal Equinox.
- the longitudes are with respect to the equatorial plane, not the ecliptic (the plane of Earth's revolution about the Sun).
- a pinnule is a sighting device used to align measurement instruments with with a target celestial body.
- Several measurements were taken as a means of verification.
- The recorded difference appears to be an average of difference measurements.

The longitudes are suspect. These are most likely calculated and rely upon geocentric models of solar motion. Kepler would not have used these values. However, because all longitudes are subject to the same constant error, the equatorial differences (in equatorial longitude) are quite accurate. Kepler had to cleanse the data and transform the equatorial longitudes to ecliptic longitudes.

### Define a metric that quantifies the error between model predictions and observed outputs.

Kepler's measurement of error was the maximal deviation from Tycho's observations. Kepler rejects his Ptolemaic circle because of an eight minute deviation between the model and Tycho's data. Clearly eight minutes was beyond Kepler's error threshold. Analysis of Kepler's works show that Kepler's model's were within two minutes of Tycho's data. That along with estimates that Tycho's data was typically accurate to within two minutes suggests that Kepler's error tolerance was two minutes.

[11] Spica is the brightest star in the constellation Virgo. Its visibility and its position very close to the ecliptic made it an attractive reference star for past astronomers.

**Apply an optimization routine to adjust the parameters and minimize the error.**

Central to Kepler's accuracy was the orientation of the ellipse and locating the Sun on the major axis. Orientation required identification of the direction for the aphelion and perihelion. Kepler's distance approximations may have been helpful. In addition, in accordance with the Second Law, the speed of Mars is slowest at the apogee and fastest at the perigee. Tycho's timestamps and directions as determined by the longitudinal difference measurements may have helped to locate where the highest and lowest speeds occur.

It is our belief that Kepler then improved the accuracy of an initial guess based upon the ideas of the previous paragraph using an iterative method with the Second Law.

**Validate results against additional data.**

Tycho made a sufficient number of observations across the planets for Kepler to both calibrate his models and then validate the outcomes on separate data. In *New Astronomy* Kepler applies this validation process to his Ptolemaic circle and upon discovering an eight minute error rejects the model. The accuracy of the *Rudolphine Tables* suggest that he applies equally strict standards in that publication.

## 5.7 Final Thoughts

As with his predecessors, Kepler applied the rigorous data driven process that is familiar to modern day data scientists. His work culminates in the *Rudolphine Tables* which proved to be the highly accurate descriptions of planetary motion; they were by far the most accurate descriptions of their day. The accuracy of subsequent forecasts of planetary positions that others derived using data from the *Rudolphine Tables* and the forecasting recipes that Kepler describes converted the European community of astronomers to Keplerian astronomy. Isaac Newton was among the converted. Newton's proof of the elliptic orbit of planets using the inverse square law rested upon the achievements of Kepler.

From the perspective of a data scientist, Kepler faced the same conundrum as Guo Shoujing. How does one select the parameters in a parametric model so that the model fits a set of observations? Kepler's solution to the conundrum is similar to the approach that Guo Shoujing most likely applied. He solved for the parameters over many subsets of observations. From these solutions, he either chose the ones that appear to best fit the data or made small adjustments to improve the fit with the data.

Kepler's problem differs from Guo Shoujing's. Guo Shoujing's parametric model is a model of convenience that does not capture any physical properties. The cubic equation is chosen because of its flexibility; with correct parameter choices it fits the data well. This type of model is known as a spline. Kepler bases his model on physical reasoning and the Second Law. [12] Kepler aligns physical reasoning with data from observations. Whether trying to parametrize a spline, or trying to parametrize a model based on physical reasoning, the need to move beyond Guo Shoujing and Kepler's ad hoc approach and formalize a process for fitting model parameters to data was on clear display. Gauss and Legendre took on the challenge.

## 5.8 Summary Poem: Kepler's Code

In castles cold and skies so wide,
Where golden noses gleamed with pride,
A noble watched the heavens turn,

[12] Isaac Newton would provide a more complete description which reduces Kepler's Second Law to a consequence of Newton's laws of motion.

While common Kepler fought to learn.

One lived in wealth, a moose for sport,
The other in a poorer court.
But fate conspired to draw them near—
The prince of stars, the seer so clear.

A clash of tempers, noble fire,
But Tycho knew what stars require.
He handed down his life's great yield—
The truths his measurements concealed.

With God and math as guiding light,
Kepler worked through day and night.
Mars, elusive, mocked his gaze,
And circled wrong through countless phase.

Eight minutes off would ruin all,
Two laws emerged, to lead the call.
He stared it down and scrapped his frame,
And changed the course of skies—and fame.

No circle's path would suit the scheme,
But ellipses, born of deeper dream.
With faith in form, and eyes precise,
He swept out areas equal—twice.

He gathered data, cleaned it well,
Transformed it, matched it, let it tell.
He tuned his laws to minimize
The gap where theory meets the skies.

A loop of steps, from guess to proof,
He iterated under truth.
A timeless script, with stars as code—
He walked the path that we call road.

A path of war, of loss, of flame,
Yet history remembers Kepler's name.
For though his life was torn apart,
He held the cosmos in his heart.

Not for reward or public cheer,
But for the truth he held most dear.
His tables spoke what stars would say,
And pointed Newton on his way.